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GAME-THEORETIC SOLUTION OF BACCARAT

JOHN G. KEMENY AND J. LAURIE SNELL, Dartmouth College

The famous gambling game of baccarat has been the subject of many mathematical studies. There is, however, considerable variation in the solutions offered and it is now clear that these variations have come from trying to solve a game of strategy without a precise meaning for the solution of such a game. The theory of games has made this precise; it is now interesting to carry out the solution of baccarat in terms of the modern concepts of game theory and to compare the solution obtained with the earlier attempts based upon the more vague ideas previously available.

Baccarat is usually played by three men, one banker playing against a pair of players. But there is a popular variant of the game, known as *chemin de fer*, in which the banker plays against a single player. This is the version commonly discussed in the literature, and we too will take this two-person version of baccarat as our subject.

1. Description of the game. The game to be considered is a card game with the special feature that hands are evaluated modulo 10. A hand will consist of two or three cards. Each card from ace through 9 is worth its face value, while each card from 10 through king is worth 10 points (and hence 0 modulo 10). Thus a hand consisting of an 8 and a 3 gives a count of 1. A count of 9 is the best possible hand.

The banker serves as dealer, dealing his opponent and himself two cards each, face down. If either man has a count of 8 or 9, he announces this fact, and hands are compared at once.

If neither player has a count of 8 or 9, then the player has the option of taking an additional card. If he elects to do so, this card is dealt him *face up*. The banker may then decide to take an additional card, if he wishes one. The hands are then compared.

When hands are compared, the higher count (modulo 10) wins. If the two men have equal counts, the game is declared a draw.

Since nonplayers may bet on the player's hand, the player's strategy is restricted by the rules of the game. He must draw if he has a count of 4 or less, and is not allowed to draw on a count of 6 or more. His only free choice arises when he has a count 5. The banker is free to make his own decisions.

2. Strategies for the player and the banker. The player has just two pure strategies: One in which he draws if dealt a 5, and one in which he does not draw in this case. The banker, on the other hand, has a wealth of strategies. He is free to decide whether to draw or not (assuming that neither player has an 8 or 9 dealt to him), knowing what cards were dealt to him, whether the player drew, and knowing what he drew. His hand may have a count from 0 to 7. The player may not have drawn, or may have drawn a card of value 0 through 9. Thus the banker has 88 possible situations in which he must decide whether to

draw. He has 2^{88} pure strategies. We are thus confronted with a 2 by 2^{88} game.

3. Reduction to a 2 by 16 game. As could be guessed from game theory, most of the strategies of the banker are dominated. Instead of pure strategies, we will discuss the 88 decisions. (A pure strategy P is said to be dominated by the strategy Q if Q is at least as good as P against all strategies of the opponent, and is sometimes better. Clearly, if Q dominates P then we may omit P from our considerations, since any time that P is used we would do as well or better to use Q.)

PLAYER'S DRAW

	None	0	1	2	3	4	5	6	7	8	9
Player does not draw on 5	5	3	3	4	4	4	5	6	6	2	2
Player does draw on 5	6	3	4	4	4	5	5	6	6	2	3

Knowing the player's strategy, the banker should draw if his count is less than or equal to the entry in this table.

FIG. 1

If, in a situation calling for a decision on the part of the banker, it is better to draw (or not to draw) against both strategies of the player, this alternative need not be considered further. Thus our first step is to compute the best choice for the banker, in each situation he might face, assuming that he knows the strategy for the player. This involves some rather complicated conditional expectation computations. The results are shown in Figure 1.

It is seen from this figure that all but four of the alternatives have the property that the best choice for the banker is the same against either strategy of the player. Thus we need only consider the banker's strategy as it relates to these four alternatives. They are:

A. The player does not draw and the banker is dealt a count of 6.
B. The player draws a 1 and the banker is dealt a count of 4.
C. The player draws a 4 and the banker is dealt a count of 5.
D. The player draws a 9 and the banker is dealt a count of 3.

There are then $2^4 = 16$ strategies which need to be considered for the banker, corresponding to possible choices of drawing or not drawing in situations *A*, *B*, *C*, and *D*.

The game has thus been reduced to a 2 by 16 game and all that remains is to compute the payoff matrix and solve the game. The payoff matrix is shown in Figure 2. The entries represent payments from the banker to the player.

4. Solution of the 2 by 16 game. We denote the banker's strategy by *NNNN*, *NNND*, *NNDN*, *etc.* We first look for dominances among his 16 strategies. We find that the strategy *NNDD* dominates *NDNN*, *NDND*, and *NDDN*. The

GAME MATRIX

Banker on *A*	*N*								*D*							
Banker on *B*	*N*				*D*				*N*				*D*			
Banker on *C*	*N*		*D*		*N*		*D*		*N*		*D*		*N*		*D*	
Banker on *D*	*N*	*D*	*N*	*D*	*N*	*D*	*N*	*D*	*N*	*D*	*N*	*D*	*N*	*D*	*N*	*D*
Player does not draw on 5	−4636	−4565	−4635	−4564	−4529	−4458	−4528	−4457	−2764	−2693	−2763	−2692	−2657	−2586	−2656	−2585
Player does draw on 5	−3585	−3690	−3600	−3705	−3590	−3695	−3605	−3710	−4001	−4106	−4016	−4121	−4006	−4111	−4021	−4126

All entries must be multiplied by 16/4826809. *N* means "does not draw," *D* means "draws."

FIG. 2

strategy *DNDD* dominates the strategies *DDNN*, *DDND*, *DDDN*. A mixture of *NNDD* and *DNDD* dominates *NDDD*, *DNNN*, *DNND*, and *DNDN*. Finally a mixture of *NNNN* and *NNDD* dominates *NNND*. This reduces the game to a 2 by 5 game. By a theorem of game theory, there is at least one 2 by 2 subgame (called a kernel) such that the solution of this 2 by 2 game gives the solution of the 2 by 5 game. Trying the 2 by 2 subgames, we find that the unique kernel is the game given by the strategies *NNDD* and *DNDD* for the banker.

The optimal mixed strategy for the player is (2/11, 9/11). The banker's mixed strategy is (1429/2288, 859/2288). The value of the game is $-679568/53094899$ or approximately -0.0128.

The banker's optimal strategy means that (1) in position A the banker should draw with probability 859/2288, (2) in position B he should not draw, (3) in C and D he should always draw.

5. Comparison with previous solutions. There have been many different "mathematically best" solutions to the game we have considered. Besides the obvious difficulty of not having a precise definition of solution, the early attempts often suffered from using heuristic ideas of expectations and conditional expectations.

It is interesting to observe that the results of modern game theory tell us the form the answer is likely to take without carrying out any computations. Our game is a 2 by N game. As such, its kernel must be 1 by 1 or at most 2 by 2. In the former case the game is strictly determined, and neither participant has a choice. But if the player is to use a mixture, as most writers correctly guessed, then the banker must mix exactly two strategies. The only alternative to this analysis would be having more than one kernel—but with the size of the numbers appearing, this is *a priori* most implausible.

As an example of a previous solution, we can find in the *Encyclopaedia Britannica* that the player (1) should not draw in A or B, (2) he should draw in C, (3) he has an "option" in D. This gives a mixture of *NNDN* and *NNDD*. While the proportions are not specified, no mixture of this kind is optimal. For example, a mixture of (1/2, 1/2) will result in an expected decrease of more than 5 per cent in the value for the banker. We also find the amazing statement: "The banker's advantage cannot be exactly calculated, but has been estimated to be 7 per cent of all the money bet against it." This is surprising, both in its negative attitude and in the much too high value attributed to the game for the banker.

A second "best" solution is given by M. Boll (*La Chance et Les Jeux de Hasard*, Paris, 1936). In this work there is a very complete and correct treatment of the conditional expectations needed for solving the game. However, having found these, the author argues as follows: The player must disguise his intention to draw or not to draw on a 5. The banker wishes to guess the player's intention. There are four possibilities corresponding to the guess of the banker and whether or not he is right. If the players are equally matched, these four will be equally likely. This leads him to the strategy (1/2, 1/2) for the player

and the pure strategy *NNDD* for the banker. This strategy is optimal against the mixed strategy (1/2/, 1/2) for the player. He arrives at a value of -0.0137 for the game. The nonoptimal mixed strategy thus increases the player's expected loss by about 7 per cent.

A more detailed presentation of these results as well as an indication of the methods used will appear in a forthcoming publication of the Committee on the Undergraduate Program of the Mathematical Association of America.

ON THE CAUCHY CRITERION FOR THE CONVERGENCE OF AN INFINITE SERIES*

ALBERT WILANSKY, Lehigh University

1. The general principle of convergence for a sequence $x=\{x_n\}$, $n=1, 2, \cdots$, of real numbers is equivalent to the following: *x is convergent if and only if for each pair of increasing sequences $\{p_n\}$, $\{q_n\}$ of positive integers with $p_n \neq q_n$ for each n, we have $x_{p_n}-x_{q_n}\to 0$ as $n\to\infty$.*

Let A^{pq} be a matrix whose nth row, for each $n=1, 2, \cdots$, consists entirely of 0's except that in the p_n place there is 1 and in the q_n place, -1. Then if a sequence x is written as a column vector, the matrix product $A^{pq}x$ is the column vector with $x_{p_n}-x_{q_n}$ as its nth entry, for each n.

The general principle of convergence then asserts that x is convergent if and only if $A^{pq}x$ is null (*i.e.* converges to 0) for each A^{pq}. The number of such A^{pq} is uncountable and the result to be proved here is that no countable such set could be sufficient for convergence. To phrase this accurately and more generally: let a *Cauchy matrix* be a matrix A with the property that Ax is null whenever x is a convergent sequence (=column vector). Each A^{pq} is a Cauchy matrix. Let a sequence x be said to *satisfy a Cauchy condition* if, for some Cauchy matrix A, Ax is null.

THEOREM. *No countable collection of Cauchy conditions is sufficient for convergence; more precisely, given a countable collection of Cauchy conditions, there exists an unbounded sequence which satisfies them all.*

2. This theorem was given by S. Mazur and W. Orlicz [1] as a paraphrase of a special result in the theory of summability; it occurs as 4.3 on page 158 of [1]. We shall give an indication of its proof. The theorem was originally stated by Mazur and Orlicz but the first published proofs are due to Karl Zeller [2, 3]; one by classical methods, inspired by ideas of R. P. Agnew; the other by means of functional analysis. Many proofs are now known.

3. The first proof of the theorem is computational and fairly long. One first proves that given a Cauchy matrix A, there exists an increasing sequence

* Invited address at the meeting of the Philadelphia Section of the Mathematical Association of America, November 24, 1956.

$r=\{r_n\}$ of positive integers such that a sequence x must obey the Cauchy condition corresponding to A if it satisfies

$$(*) \qquad \lim_n \max_{r_n \le u, v \le r_{n+1}} |x_u - x_v| = 0.$$

Corresponding to a sequence of Cauchy matrices we get a sequence $\{r^k\}$ of sequences. There exists a single sequence r such that any x satisfying $(*)$ must satisfy each condition obtained from $(*)$ by replacing r by r^k. Since there is clearly an unbounded sequence satisfying $(*)$, this concludes the proof. (For details see [3], pp. 140, 141, and §12, p. 150.)

4. The second proof depends on deeper ideas but has the advantages of brevity and intrinsic interest.

5. Given a Cauchy matrix A, let $\mathfrak{A}$ be the set of all sequences obeying the corresponding Cauchy condition *i.e.* such that Ax is null. It is possible—the exact details can be omitted—to introduce a definition of distance between two elements of $\mathfrak{A}$ such that the various following statements are correct. See [2] for details. Given any linear and continuous real function f defined on $\mathfrak{A}$, there exists a sequence $\{b_k\}$ of real numbers such that for each bounded sequence $x \in \mathfrak{A}$ we have $f(x)=\sum b_n x_n$. (See [2], pp. 476, 479.) In the following lemma, c_0 is the subset of $\mathfrak{A}$ consisting of the null sequences.

LEMMA 1. *If such a function f vanishes on c_0, then $f(x)=0$ for each bounded $x \in \mathfrak{A}$*

Proof. As above, $f(x)=\sum b_n x_n$. Now suppose that f vanishes on c_0. Let, for $k=1, 2, \cdots$, δ^k be the sequence of 0's save for 1 in the kth place. Then $0=f(\delta^k)=b_k$ for each k and so $f(x)=0$.

LEMMA 2. *Given a Cauchy condition, at least one unbounded sequence must satisfy it.*

This means that if A is a Cauchy matrix, Ax is null for at least one unbounded x. Assume the contrary. Then $\mathfrak{A}$ contains only bounded sequences. We shall deduce from this the false conclusion that $\mathfrak{A}$ contains only null sequences, contradicting the definition of a Cauchy matrix. Let $L_n(x)=x_n$, $n=1, 2, \cdots$; this defines L_n for each n, a linear and continuous real function on $\mathfrak{A}$. Our hypothesis is that $\{L_n(x)\}$ is bounded for each x. By the uniform boundedness principle $\{L_n\}$ is an equicontinuous family, *i.e.* given $\epsilon>0$, there exists $\delta>0$ such that $|x_n-y_n|<\epsilon$ for each n if the distance between x and y is less than δ.

For arbitrary $x \in \mathfrak{A}$, let $\epsilon>0$ be given, and choose $y \in c_0$ with $|x_n-y_n|<\epsilon$ for each n. This can be done since, by Lemma 1, c_0 is dense in $\mathfrak{A}$. Then $|x_n| \le |y_n| + |x_n-y_n| < |y_n| + \epsilon$. Hence lim sup $|x_n| \le \epsilon$ and so x is null.

This computation is simply the verification that an equicontinuous family which is convergent on a dense set is convergent everywhere.

6. We now turn to the proof of the main theorem. Given a sequence of Cauchy conditions, the set of sequences satisfying all these conditions is the intersection of the sets satisfying each one, hence the intersection of a sequence of metric spaces of the type we are considering. Such a space can again be given a metric of the same type: specifically, each metric is given by a sequence of seminorms and we get the metric for the intersection by using all the seminorms. (For details see [2], 4.7, p. 472.)

On the intersection space we again have the same representation of a linear and continuous real function as is mentioned above before Lemma 1. Hence the proofs of Lemmas 1 and 2 proceed unchanged and we obtain our result.

References*

1. S. Mazur and W. Orlicz, On linear methods of summability, Studia Math., vol. 14, 1954, pp. 129–160 [16, 814].
2. Karl Zeller, Allgemeine Eigenschaften von Limitierungsverfahren, Math. Z., vol. 53, 1951, pp. 463–487 [12, 604].
3. Karl Zeller, Faktorfolgen bei Limitierungsverfahren, Math. Z., vol. 56, 1952, pp. 134–151 [14, 158].

* The numbers in square brackets refer to *Mathematical Reviews*, volume and page.

MÖBIUS TETRADS*

SAHIB RAM MANDAN, Indian Institute of Technology, Kharagpur, India

Introduction. An incomplete model of a pair of Möbius tetrads served as a hanger until the following chain of observations made its construction simple and led to the very interesting results enumerated below. Excepting the existence of the Möbius tetrads explained in Baker's *Principles of Geometry* (vol. 3, pp. 67–68, 138), all the results established here are original.

(i) A quadric Q_1 is discovered interlocked with a pair of Möbius tetrads T_0 and T_1 that leads to the construction of infinitely many mutually interlocked tetrads all further interlocked with Q_1. Sets of four pairs of Möbius tetrads are observed, each set having the same eight vertices.

(ii) T_0 and T_1 are harmonic inverses of each other with respect to a pair of skew generators of the quadric Q for which T_0, T_1, and Q_1 are self-polar. These generators form a pair of polar lines for Q_1, and Q and Q_1 have a pair of skew generators in common.

(iii) There are two pairs of polar lines for Q_1 upon which lie the vertices, one on each, of a third tetrad T_2 that is interlocked with T_0 and T_1 independently of the infinity of tetrads in (i). Given one vertex, T_2 is uniquely determined.

* Read at the 42nd session of I.S.C.A., Baroda, January, 1955.

We proceed to establish these results in the following sections.

1. Existence.

1.1 Discovery of a quadric. Let $T_0 = ABCD$ and $T_1 = A_1B_1C_1D_1$ be a pair of Möbius tetrads such that $ABCD_1$, $BCDA_1$, $CDAB_1$, $DABC_1$; $A_1B_1C_1D$, $B_1C_1D_1A$, $C_1D_1A_1B$, $D_1A_1B_1C$ are the eight coplanar tetrads that constitute the faces of T_0 and T_1. This arrangement suggests that AB_1, A_1B, CD_1, C_1D are a set of four skew lines that intersect another set, *viz.*, AB, CD, A_1B_1, C_1D_1, thus forming the two systems of generators of a quadric Q_1 evidently inscribed as well as circumscribed to both T_0 and T_1, *i.e.*, T_0 and T_1 are m.i.* For, any face of either tetrad contains a pair of generators of the quadric and hence is a tangent plane, as is evident from Figure 1. Evidently, there are three quadrics of the type Q_1, each

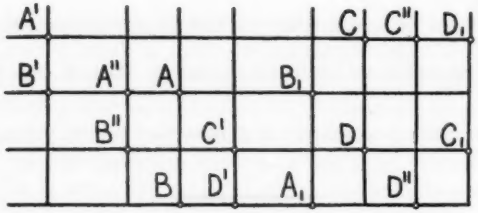

FIG. 1

having a pair of opposite edges, of each tetrad, as generators, thus proving that the vertices of the two Möbius tetrads form a set of eight associated points.

1.2 Infinity of tetrads. If we take any two generators of the second system of the preceding paragraph, one intersecting any two lines of the first set in A', B' and the other intersecting the other two in C', D', we find that $A'B'C'D'$ is a tetrad T' that is interlocked with Q_1, and in an order with T_0 and T_1 (Fig. 1.) Again, if we construct another tetrad T'' in the same manner as T', we easily see that Q_1, T', T'', T_0, and T_1 are all m.i., suggesting thus the construction of infinitely many m.i. tetrads all interlocked with Q_1.

Incidentally, we may now observe the three pairs of Möbius tetrads ABC_1D_1, A_1B_1CD; AB_1CD_1, A_1BC_1D; AB_1C_1D, A_1BCD_1 that, together with T_0 and T_1 form a set of four pairs of Möbius tetrads having the same eight vertices. Any pair gives rise to such a set.

1.3 Equation of the quadric. If the coordinates of the vertices of T_1 referred to T_0 be written† [B-3, p. 138] as $(0, -n_1, m_1, -l_1)$, $(n_1, 0, -l_1, -m_1)$, $(-m_1, l_1, 0, -n_1)$, $(l_1, m_1, n_1, 0)$, the equation of Q_1 is found to be $l_1(yz+tx) + m_1(yt-zx) = 0$, whose generators are

---

* m.i. denotes mutually interlocked.

† B-i denotes H. F. Baker, Principles of Geometry, vol. i (i = 1, 2, 3, 4).

$$x - ky = 0 = l_1z + m_1t - k(m_1z - l_1t),$$

$$x - k'(l_1z + m_1t) = 0 = y - k'(m_1z - l_1t),$$

k and k' being the parameters. The lines of the first set mentioned in Section 1.1 are given by $k = \infty$, 0, l_1/m_1, $-m_1/l_1$, and those of the second set by $k' = \infty$, 0, $-n_1/(l_1^2+m_1^2)$, $1/n_1$.

2. Self-polar quadric.

2.1 Transversals. Let U, U' and V, V' be pairs of conjugate points for Q_1 of Section 1.1 on AA_1 and BB_1, respectively. Since AA_1 and BB_1 form a pair of polar lines for Q_1, UV and $U'V'$ will also be a pair of polar lines for Q_1 (Fig. 2). Again,

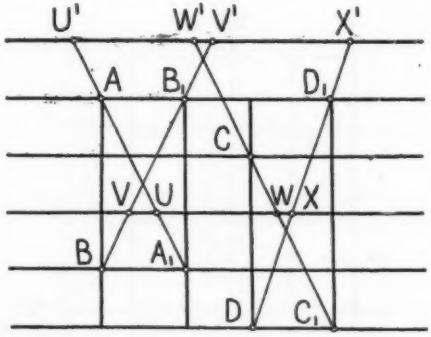

FIG. 2

if UV meets CC_1 in W, and W' be on CC_1 conjugate to W for Q_1, $U'V'W'$ will be the polar plane of W for Q_1 that must pass through the polar line DD_1 [B-3, p. 67] of CC_1 for Q_1. Further, if UVW meets DD_1 also, say in X, and X' be on DD_1 conjugate to X for Q_1, then U', V', W', X' become collinear in the polar line of $UVWX$ for Q_1. This proves that the two transversals of the four lines, AA_1, BB_1, CC_1, DD_1, joining the corresponding vertices of a pair of Möbius tetrads, T_0 and T_1, constitute a pair of polar lines for Q_1.

2.2 Harmonic inversion with respect to a pair of skew lines. Two points, B, B_1, are harmonic inverses of each other with respect to skew lines x, y, if BB_1 meets the lines in points B', B'', that separate B, B_1 harmonically. If C, C_1 are also harmonic inverses with respect to x, y, the lines BC and B_1C_1 are said to be harmonic inverses of each other with respect to x, y, for any point on BC is the inverse of a point on B_1C_1 and conversely, and the cross-ratio is unaltered by this inversion [B-3, p. 3]. If D, D_1 is a third pair of points that are harmonic inverses with respect to x, y and neither collinear with BC nor with B_1C_1, the planes BCD, $B_1C_1D_1$ are said to be harmonic inverses with respect to x, y. The line l joining the meets, L', L'', of x, y with either plane, say BCD, is its own

inverse and the other plane, $B_1C_1D_1$, is none other than the fourth harmonic of the first plane with respect to the planes lx, ly (Fig. 3). If x, y happen to be a pair of generators of the same system of a quadric Q, it is easy to see that any point has for its inverse with respect to x, y, its conjugate with respect to Q. A pair of conjugate points or planes for Q has for its inverse a pair of conjugate points or planes for Q (cross-ratio property). If x, y is a pair of polar lines for a quadric, the quadric is its own harmonic inverse.

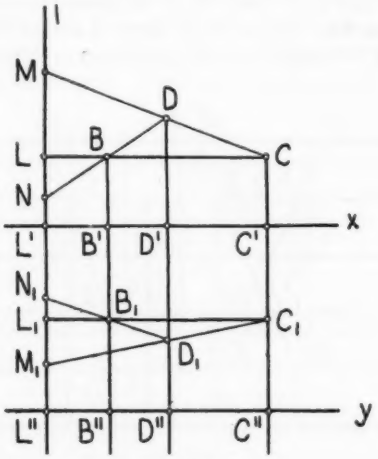

FIG. 3

We may now observe that a pair of Möbius tetrads, T_0, T_1, are harmonic inverses of each other (Fig. 2) with respect to the transversals (Sec. 2.1) $x \equiv UVWX$, $y \equiv U'V'W'X'$, of the four joins of the corresponding vertices of the tetrads [**B-3**, p. 144], and Q_1 is its own inverse.

2.3 Construction. (a) Let us construct a quadric Q having $x \equiv UVWX$, $y \equiv U'V'W'X'$ as defined in Section 2.1 as a pair of skew generators and the triangle BCD self-conjugate for Q. Since AA_1, BB_1, CC_1, DD_1, are pairs of harmonic inverses with respect to x, y (Fig. 2), they are evidently pairs of conjugate points for Q. The polar planes of B, C, D are then CDB_1, BDC_1, BCD_1, respectively, all containing A (Sec. 1.1), proving that BCD is the polar plane of A, *i.e.*, T_0 is self-polar for Q.

(b) Since T_1 is the harmonic inverse of T_0 with respect to a pair of generators of Q, it must also be self-polar for Q (Sec. 2.2; [**B-3**, p. 68, Ex. 22]). Hence, the pair of transversals to the four lines joining the corresponding vertices of a pair of Möbius tetrads are generators of the quadric for which the given tetrads are self-polar. It is interesting to observe that the polar reciprocal of Q_1 (Sec. 1.1)

with respect to Q is Q_1 itself. This follows since the polar reciprocal, being defined as the locus of the poles of tangent planes of Q_1 as well as the envelope of the polar planes of points of Q_1 with respect to Q, must be Q_1 itself, since it passes through seven independent vertices and touches seven independent faces of the tetrads, T_0, T_1, which are poles and polar planes, respectively, of seven tangent planes and seven points of Q_1 with respect to Q. Hence, Q_1 is also self-reciprocal or self-polar for Q.

2.4 Generators common to two quadrics. Let us now construct the pair of generators u, v, of the quadric Q_1 (Sec. 1.1) that separate harmonically the two pairs of its generators of the first system, AB_1, A_1B and CD_1, C_1D, which we

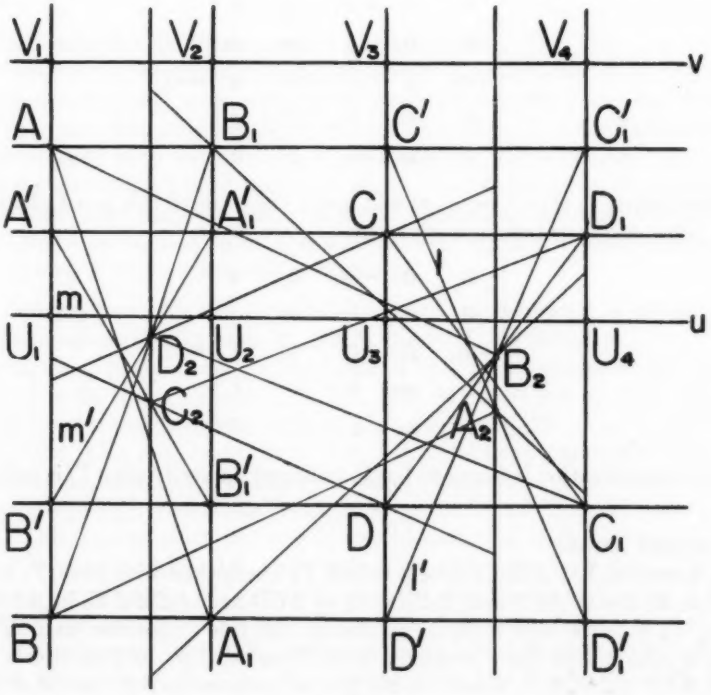

FIG. 4

notice to be the pairs of polar lines for Q. This is true since, by the preceding section, the polar planes A_1BCD and $A_1BC_1D_1$ of A, B_1 with respect to Q have the line A_1B in common, and the polar planes C_1DAB, $C_1DA_1B_1$ of C,D_1 with respect to Q have the line C_1D in common. From the nature of the construction of u, v we see that U_i, V_i $(i=1, 2, 3, 4)$ on u, v, respectively separate harmonically the two pairs of conjugate points for Q on each of the generators AB, A_1B_1, CD,

C_1D_1 (Fig. 4), of the second system of Q_1 where they meet the two pairs of polar lines, AB_1, A_1B and CD_1, C_1D for Q; that is U_i, V_i are the united elements of the involution set up by the two pairs of conjugate points for Q on the respective lines and hence are none other than the pairs of intersections of these lines with Q. In other words, we prove that they lie on Q, showing that u, v are the common generators of Q and Q_1.

2.5 Equations. The equation of Q referred to T_0 is found to be $x^2+y^2+z^2+t^2=0$, and u, v are given by (Sec. 1.3) $k=\pm i$, that is, $x\pm iy=0=z\mp it$. To find the polar reciprocal of Q_1 with respect to Q, let $lx+my+nz+pt=0$ be a tangent plane of Q_1. The condition of tangency implies [**B-3**, p. 24]

$$\begin{vmatrix} 0 & 0 & -m_1 & l_1 & l \\ 0 & 0 & l_1 & m_1 & m \\ -m_1 & l_1 & 0 & 0 & n \\ l_1 & m_1 & 0 & 0 & p \\ l & m & n & p & 0 \end{vmatrix} = 0.$$

Its pole referred to Q is (l, m, n, p) and hence the locus of this point, defined to be the polar reciprocal of Q_1, is given by

$$\begin{vmatrix} 0 & 0 & -m_1 & l_1 & x \\ 0 & 0 & l_1 & m_1 & y \\ -m_1 & l_1 & 0 & 0 & z \\ l_1 & m_1 & 0 & 0 & t \\ x & y & z & t & 0 \end{vmatrix} = 0,$$

which on simplification is found to be Q_1. It is easy to verify that T_1 is self-polar for Q.

3. A third tetrad.

3.1 Location. Let $A_2B_2C_2D_2$ be a tetrad T_2 interlocked with both T_0 and T_1 such that A_2 lies in BCD and $B_1C_1D_1$, B_2 in ACD and $A_1C_1D_1$, C_2 in ABD and $A_1B_1D_1$, D_2 in ABC and $A_1B_1C_1$. Evidently, the line l common to BCD and $B_1C_1D_1$ is polar to the line l' common to ACD and $A_1C_1D_1$, and the line m common to ABD and $A_1B_1D_1$ is polar to the line m' common to ABC and $A_1B_1C_1$ for Q_1 (Fig. 4). Hence, the vertices of T_2, independently of the infinity of m.i. tetrads (Sec. 1), lie on two pairs of polar lines for Q_1.

3.2 Three quadrics. If T_2 is interlocked with T_0 and T_1 as described in the preceding paragraph, then, following Section 1, AB, A_2B_2, CD, C_2D_2, with AB_2, A_2B, CD_2, C_2D, will generate a quadric Q_2, and A_1B_1 A_2B_2, C_1D_1, C_2D_2, with A_1B_2, A_2B_1, C_1D_2, C_2D_1, generate a quadric Q_{12}. The three quadrics Q_1, Q_2, Q_{12}, have a pair of skew generators, say u', v', in common. This follows since Q_1 and

Q_2, having a pair of skew generators AB, CD in common, will have another pair of generators of the other system in common, which will therefore intersect all the generators of the first system and, in particular, A_1B_1, C_1D_1, of Q_1 and A_2B_2, C_2D_2, of Q_2. Hence, they will be generators of Q_{12} also.

To locate these generators, we notice that A_2, B_2 and C_2, D_2, being pairs of conjugate points for Q_1 (Sec. 3.1), A_2B_2, C_2D_2 will meet Q_1 in pairs of points, conjugate for Q_2 and Q_{12}, that must lie on u', v', and which will therefore separate harmonically (Fig. 4) the pairs of generators of the same system of both Q_2 and Q_{12} through A_2, B_2 and C_2, D_2, respectively. Then A, B; C, D; A_1, B_1; C_1, D_1 will be pairs of harmonically inverse points with respect to u', v' (following the property of related ranges, [**B-3**, p. 3]), thus identifying u', v' with u, v of Section 2.4.

3.3 Self-polar for Q. Since u, v are generators of Q (Sec. 2.3), A_2, B_2 and C_2, D_2 form pairs of conjugate points for Q also. Furthermore, A_2 is conjugate to A as well as A_1 for Q, since it lies in their polar planes BCD and $B_1C_1D_1$ with respect to Q, and the plane $B_2C_2D_2$ contains A as well as A_1, since T_2 is assumed to be interlocked with both T_0 and T_1. We can say, therefore, that the polar plane of A_2 with respect to Q, passing through three of its conjugate points, B_2, A, A_1, is none other than the plane $B_2C_2D_2$. Similarly, it can be proved that the polar planes of B_2, C_2, and D_2 for Q will be $A_2C_2D_2$, $A_2B_2D_2$, and $A_2B_2C_2$, respectively, showing that T_2 is also self-polar for Q.

3.4 Construction. We are now in a position to construct a tetrad $T_2 \equiv A_2B_2C_2D_2$,interlocked with both T_0 and T_1 (Sec. 1.1), as follows:

Through any point A_2 on the line l (Sec. 3.1) draw a transversal to u, v that will meet l' in B_2. Then, l, l', u, v, CC_1, DD_1 (Fig. 4) generate a quadric following the property of related ranges on CD and C_1D_1 [**B-3**, p. 3]. The quadric generated by m, m', u, v, AA_1, BB_1, and that by AB, CD, A_2B_2, having the generators AB, u, v in common, must have another generator in common that meets m, m' in C_2, D_2, respectively. Then $A_2B_2C_2D_2$ is the required tetrad T_2 that is interlocked with T_0. This follows, since A_2 lying in BCD means that A_2B intersects CD, so that A_2B intersects A_2B_2, AB, and hence C_2D_2 also. The lines AB_2, CD_2, C_2D have a similar property, thus determining the quadric Q_2 of Section 3.2. To show that T_2 is also embedded with T_1, we observe that C_2D_2 intersects AA_1 as well as AB_2, showing that A_1B_2 intersects C_2D_2; but A_1B_2 intersects A_1B_1 and C_1D_1 also. Similarly, A_2B_1, C_1D_2, C_2D_1 intersect the four lines A_1B_1, A_2B_2, C_1D_1, C_2D_2, proving finally what is required. Hence, given one vertex, T_2 is uniquely determined.

4. Further results. In addition to the results (i), (ii), (iii) stated in the introduction, it is possible to prove the following:

(iv) The tetrads T_0, T_1, T_2, give rise to another tetrad T_3, forming a set of four mutually interlocked tetrads that lead to another set of four such tetrads, having the same sixteen vertices (faces) that lie (touch) by sixes on sixteen

conics (cones), one in (through) each face (vertex), having in common with one another a pair of vertices (faces as tangent planes). The eight tetrads of the two sets are all self-polar with respect to Q. There exist thirty-six quadrics out-polar (in-polar) to Q, each one passing through (enveloping) four conics (cones) and twelve vertices (faces) that form two pairs of Möbius tetrads, one pair from each set. Nine quadrics pass through (envelop) each conic (cone).

(v) There are eighteen quadrics Q_i for the set of tetrads T_i ($i=0, 1, 2, 3$), three for each pair, divided into three sets of six each, each set having a fixed pair of skew generators in common with Q. The same is the case with the second set of tetrads. The relations of these thirty-six quadrics with those of (iv) are quite remarkable.

Proofs of these results will appear later.

COMPLEX NUMBERS AND TRIGONOMETRY

D. E. RICHMOND, Williams College

1. Introduction. The present paper is concerned with a method of presenting trigonometry which greatly reduces the time which needs to be devoted to the subject and also contributes significantly to the mathematical maturity of the student. This method is to treat trigonometry as a branch of the theory of complex numbers. The present treatment is a development of an approach used at Williams College in order to handle in a single freshman course students who have had no preparation in trigonometry along with those who have had such a preparation. The treatment is sufficiently novel to interest those already familiar with the subject without being too difficult for beginners. The material prior to Section 6 could very well be taught in the fourth year of high school. I regard this as entirely practicable. It has the advantage of familiarizing students with complex numbers and vectors and opening the way to significant connections with other branches of mathematics.

2. Complex numbers. Our treatment of complex numbers is along conventional lines with one exception. Specifically:

(1) Two complex numbers, $\alpha=a+ib$ and $\beta=c+id$, are defined to be equal if and only if $a=c$ and $b=d$.

(2) Sums, differences and products are defined by

$$\alpha \pm \beta = (a \pm c) + i(b \pm d), \alpha\beta = (ac - bd) + i(ad + bc).$$

(3) Complex numbers are represented by vectors which are added and subtracted in the familiar way.

(4) The absolute value $|\alpha|$ of α is defined by $|\alpha|^2=a^2+b^2=\alpha\bar{\alpha}$ where $\bar{\alpha}=a-ib$.

The angle θ of α is to be measured (in degrees or radians) counterclockwise from the positive real axis.

(5) However, instead of justifying the usual geometric representation of the product of α and β by introducing the polar form and appealing to the addition formulas of trigonometry, we adopt an idea from Dresden's *An Invitation to Mathematics* (pp. 89, 90) to prove directly that when two factors are multiplied, their magnitudes are multiplied and their angles are added. The proof, which differs in detail from his, follows in Section 3.

3. Multiplication of complex numbers.

a) It is easy to verify that $\overline{\alpha\beta} = \bar{\alpha}\bar{\beta}$, that is, that the conjugate of the product is the product of the conjugates. Hence

$$|\alpha\beta|^2 = (\alpha\beta)(\overline{\alpha\beta}) = \alpha\beta\bar{\alpha}\bar{\beta} = \alpha\bar{\alpha}\beta\bar{\beta} = |\alpha|^2|\beta|^2.$$

Since magnitudes are nonnegative, $|\alpha\beta| = |\alpha| \cdot |\beta|$.

b) In discussing the angle of the product, it is sufficient to assume that $|\alpha| = |\beta| = 1$, since an alteration in the magnitudes of α and β would not change the direction of their product. Let θ and ϕ be the angles of α and β. (See Figs. 1a, 1b.) Then chord $BC = |\alpha\beta - \beta| = |(\alpha-1)\beta| = |\alpha-1|\,|\beta| = |\alpha-1|$ = chord AR. Hence $\angle BOC = \theta$ and $\angle ROC = \phi + \theta$ or $\phi - \theta$.

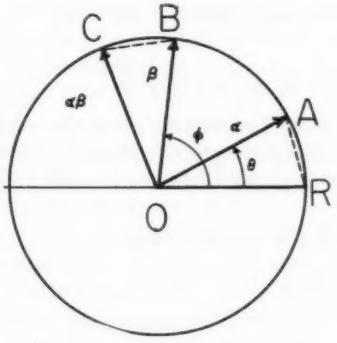

FIG. 1a

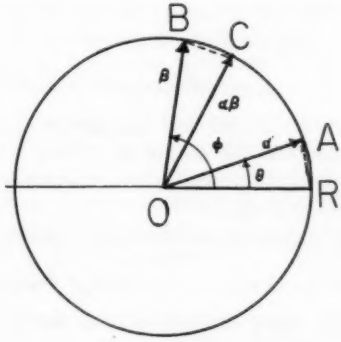

FIG. 1b

Interchanging the role of α and β in the proof gives $\angle ROC = \theta + \phi$ or $\theta - \phi$. The desired conclusion follows unless $\phi - \theta = \theta - \phi$, that is, unless $\phi - \theta = 0°$ or $180°$. For the first alternative, $\alpha = \beta$, and we are asked to believe that $\alpha^2 = 1$. This is possible only if $\alpha = \beta = 1$ or $\alpha = \beta = -1$, when addition and subtraction of angles give equivalent answers (identifying angles which differ by $360°$). The alternative $\phi = \theta + 180°$ is handled similarly.

4. The addition formulas of trigonometry. A complex number α of unit magnitude ($|\alpha| = 1$) is completely specified by the angle θ between it and the posi-

tive real axis. (We shall use degree measure.) It will be appropriate to write $\alpha=\alpha(\theta)$. We define $\cos\theta$ to be the real part of $\alpha(\theta)$ and $\sin\theta$ to be its imaginary part. (See Fig. 2.)

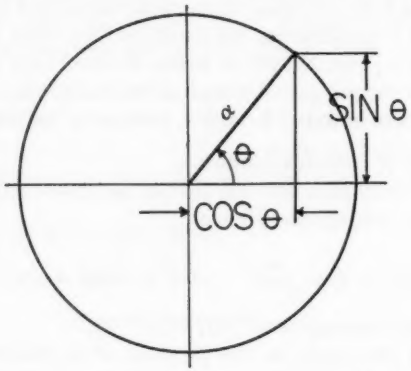

FIG. 2

Let $\alpha(\theta)$ and $\alpha(\phi)$ have unit magnitude. By definition

$$\alpha(\theta) = \cos\theta + i\sin\theta, \alpha(\phi) = \cos\phi + i\sin\phi.$$

Multiply $\alpha(\theta)$ by $\alpha(\phi)$, first by using the definition, secondly by using the theorem of Section 3 which gives

$$\alpha(\theta)\alpha(\phi) = \alpha(\theta+\phi) = \cos(\theta+\phi) + i\sin(\theta+\phi).$$

Equating the real and imaginary parts of these two results gives the addition formulas for the cosine and sine. This proof is completely general, involving no separate consideration of quadrants. It has the additional merit of being extremely simple.

To prove the subtraction formulas, multiply the equation

$$\alpha(\phi-\theta)\alpha(\theta) = \alpha(\phi)$$

by $\bar{\alpha}(\theta)$, using $\alpha(\theta)\bar{\alpha}(\theta)=1$, to obtain

$$\alpha(\phi-\theta) = \alpha(\phi)\bar{\alpha}(\theta).$$

Multiply out on the right and equate real and imaginary parts. The special case $\phi=0$ gives $\alpha(-\theta)=\bar{\alpha}(\theta)$ and hence $\cos(-\theta)=\cos\theta$, $\sin(-\theta)=-\sin\theta$.

5. Multiple angles. Calculation of tables. From the results of Section 4, it follows immediately that $[\alpha(\theta)]^n=\alpha(n\theta)$ which yields the well-known formula of de Moivre

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta,$$

and thence formulas for $\cos n\theta$ and $\sin n\theta$, in particular, for $\cos 2\theta$ and $\sin 2\theta$.

The sines and cosines of the quadrantal angles and of 30°, 45°, and 60°, with their relatives in other quadrants, are easily obtained by simple geometrical arguments. It is of interest to note how easily one may construct a complete table of sines and cosines for all angles which are integral multiples of 1°.

We may begin by calculating $\alpha=\alpha(72°)$. Clearly $\alpha^5=1$ or $\alpha^5-1=0$. Removing the uninteresting root $\alpha=1$, we have $\alpha^4+\alpha^3+\alpha^2+\alpha+1=0$. Dividing by α^2 and noting that $1/\alpha=\bar{\alpha}$,

$$\alpha^2 + \alpha + 1 + \bar{\alpha} + \bar{\alpha}^2 = 0. \tag{1}$$

Now $\alpha+\bar{\alpha}=2\cos 72°=x$, say, and squaring and simplifying, $\alpha^2+\bar{\alpha}^2=x^2-2$. Then (1) becomes $x^2+x-1=0$ whose positive root is $(\sqrt{5}-1)/2$. From $\cos 72° = (\sqrt{5}-1)/4$, we obtain successively cos 36°, cos 18°, cos 9° and sin 9°.

Similarly, $\alpha(40°)$ satisfies $\alpha^9-1=0$. The same procedure, using easily derived expressions for $\alpha^3+\bar{\alpha}^3$ and $\alpha^4+\bar{\alpha}^4$, gives $x^4+x^3-3x^2-2x+1=0$ for $x=2\cos 40°$. Removing the obvious root $x=-1$, we must solve

$$x^3 - 3x + 1 = 0 \tag{2}$$

for its largest positive root (1.53208). From cos 40°, one readily calculates cos 20°, cos 10° and sin 10°. From cos 9° and sin 9°, previously calculated, and the subtraction formulas we have sin 1°, cos 1°, and thence the sines and cosines of all integral multiples of 1°.

It is to be noted that cos 40° is not constructible by ruler and compass while cos 72° and cos 60° are so constructible. It follows that the only angles with an integral number of degrees which are constructible are multiples of 3°.

The method described furnishes a natural bridge to problems in the theory of equations. In this connection it is pertinent to show the solution of (2) by *iteration*, a method which deserves to be included in our elementary courses. We write (2) in the form $x=\sqrt[3]{3x-1}$ and substitute under the radical a guess (x_0) at the required solution, obtaining $x_1=\sqrt[3]{3x_0-1}$. Feed back x_1 under the radical to obtain $x_2=\sqrt[3]{3x_1-1}$ and continue until agreement is reached to the requisite accuracy. Concretely, if this process is applied to the present case, starting with $x_0=1.5$, one obtains from a table of cube roots, the sequence $x_1=1.5183$, $x_2=1.526$, $x_3=1.529$, $x_4=1.531^-$, $x_5=1.531^+$. Agreement to three decimal places is reached at $x=1.532$.

We may well ask for assurance that this process converges and that it yields the desired solution. Clearly

$$x_{n+1}^3 = 3x_n - 1. \tag{3}$$

Then

$$x_{n+2}^3 = 3x_{n+1} - 1, \qquad x_{n+2}^3 - x_{n+1}^3 = 3(x_{n+1} - x_n).$$

If therefore $x_{n+1}>x_n$, then $x_{n+2}>x_{n+1}$.

Since in fact $x_1>x_0$, the solutions $x_0, x_1, x_2, \cdots$ form an increasing sequence.

This sequence has 2 as an upper bound. For, if $x_{n+1}>2$, by (3) $x_n>3>2$ and it would follow that $x_0>2$, contradicting $x_0=1.5$. The sequence $\{x_n\}$ therefore converges to a limit $L\leqq 2$. It is easy to reduce this upper bound.

Since $x_n\to L$, $x_{n+1}\to L$ and from (3), $L^3=3L-1$, so that L is indeed a solution of the required equation.

6. Calculus of trigonometric functions. We now turn to those properties of the sine and cosine functions which require calculus ideas for their expression and proof. For this purpose we use radian or arc measure. Let x be the arc length measured counterclockwise from A and $\alpha(x)$ the complex number represented by the vector $\overrightarrow{OP}$ (see Fig. 3). We proceed to determine the derivative of $\alpha(x)$.

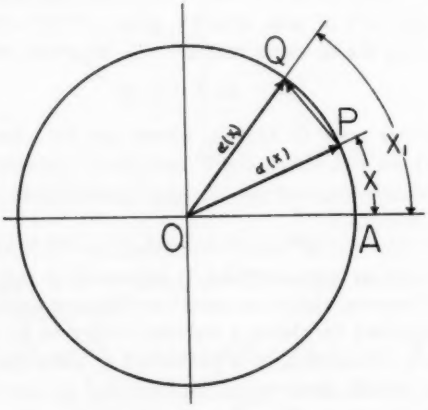

FIG. 3

Let x_1 be a second value of the arc length and $\alpha(x_1)$ the corresponding complex number. The derivative $D\alpha$ is defined to be the limit as $x_1\to x$ of the ratio $[\alpha(x_1)-\alpha(x)]/[x_1-x]$, provided that this limit exists.

Now $\alpha(x_1)-\alpha(x)$ is represented by the vector $\overrightarrow{PQ}$ along the chord of the unit circle while x_1-x is the length of the arc PQ, and $[\alpha(x_1)-\alpha(x)]/[x_1-x]$ is directed along $\overrightarrow{PQ}$. As $x_1\to x$, the ratio of the length of chord PQ to arc PQ approaches 1 and the direction approaches that of the tangent to the circle at P. Hence

$$\frac{\alpha(x_1)-\alpha(x)}{x_1-x}\to\tau(x),$$

where $\tau(x)$ is the complex number represented by the unit tangent vector PT (Fig. 4). But $\tau(x)=i\alpha(x)$, since multiplication by i rotates $\alpha(x)$ through a right

angle. Hence

$$D\alpha = i\alpha. \tag{4}$$

This result gives almost immediately the derivatives of $\sin x$ and $\cos x$, since

$$D(\cos x + i \sin x) = i(\cos x + i \sin x)$$

or

$$D \cos x + iD \sin x = i \cos x - \sin x.$$

Hence

$$D \cos x = -\sin x, \qquad D \sin x = \cos x.$$

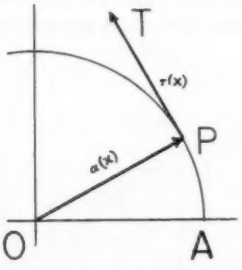

FIG. 4

Equation (4) suggests the notation e^{ix} for $\alpha(x)$ since (4) is then the analogue of $De^{ax} = ae^{ax}$. It is of course open to us to define e^{ix} to be the function previously called $\alpha(x)$, so that in this new notation

$$e^{ix} = \cos x + i \sin x,$$

the famous Euler formula. We shall not, however, adopt this notation since we wish our treatment to be independent of any prior knowledge of the exponential function.

We proceed to solve (4) for $\alpha(x)$ subject to the condition, $\alpha(0) = 1$. Integrating from 0 to x, we have the equivalent equation

$$\alpha(x) - 1 = i \int_0^x \alpha(x)dx. \tag{5}$$

We solve (5) by iteration in a form suitable for computation, starting on the right with $\alpha_0(x) = 1$ and finding in succession $\alpha_1(x)$, $\alpha_2(x)$, $\cdots$, where

$$\alpha_{n+1}(x) = 1 + i \int_0^x \alpha_n(x)dx. \tag{6}$$

We obtain

$$\alpha_1(x) = 1 + ix,$$

$$\alpha_2(x) = 1 + i\left(x + \frac{ix^2}{2}\right) = 1 + ix + \frac{(ix)^2}{2},$$

$$\alpha_3(x) = 1 + ix + \frac{(ix)^2}{2} + \frac{(ix)^3}{2.3}.$$

By induction, one easily proves that

$$\alpha_n(x) = 1 + ix + \frac{(ix)^2}{2!} + \cdots + \frac{(ix)^n}{n!}.$$

It is of interest to construct $\alpha_n(x)$ graphically. For simplicity we assume that $x<1$. This is no handicap in practice, since we require $\sin x$ and $\cos x$ (and hence $\alpha(x)$) only for $x \leqq \pi/4$.

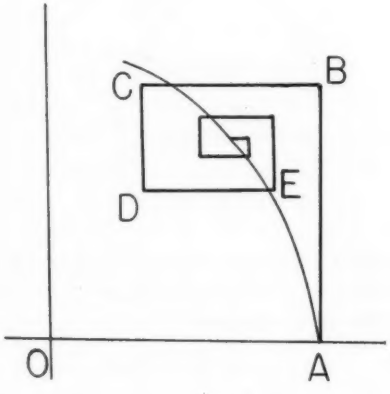

FIG. 5

We plot as points in the complex plane in succession (Fig. 5)

$$A\colon\ \alpha_0 = 1,$$

$$B\colon\ \alpha_1 = 1 + ix,$$

$$C\colon\ \alpha_2 = 1 + ix - \frac{x^2}{2!},$$

$$D\colon\ \alpha_3 = 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!},$$

$$E\colon\ \alpha_4 = 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!}.$$

Joining O to A, to B, to C, *etc.*, by straight-line segments gives a spiral. Each term, $(ix)^n/n!$, which is added, differs from its predecessor by the factor ix/n. The factor i corresponds to a counterclockwise rotation of 90°. With $x<1$, as assumed, $x/n<1$, so that the successive segments of the spiral become shorter. Moreover, $x^n/n!<1/n!<1/n$, so that the length of the nth segment approaches zero. The α_n's converge to a limit $\alpha(x)$. In fact, if $r_n(x)=\alpha(x)-\alpha_n(x)$, it is geometrically clear that $|r_n(x)|<2/n$. It remains to prove that $\alpha(x)$ is the required solution so that

$$\alpha(x) = 1 + i\int_0^x \alpha(x)dx.$$

Proof. We have from (6)

$$\alpha_{n+1}(x) = 1 + i\int_0^x \alpha_n(x)dx$$

$$= 1 + i\int_0^x \alpha(x)dx - i\int_0^x r_n(x)dx.$$

Since $\alpha_{n+1}(x)\to\alpha(x)$, (5) follows if $\int_0^x r_n(x)dx\to 0$. Now

$$\left|\int_0^x r_n(x)dx\right| \leqq \int_0^x |\,r_n(x)\,|\,dx \leqq \int_0^x \frac{2}{n}\,dx = \frac{2x}{n} \leqq \frac{2}{n},$$

so that $\int_0^x r_n(x)dx\to 0$ as $n\to\infty$.

From

$$\alpha(x) = 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \cdots$$

one obtains the well-known series for cos x and sin x by taking the real and imaginary parts, respectively. Thus

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots, \qquad \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots.$$

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THE WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

L. E. BUSH, Kent State University

The following results of the seventeenth William Lowell Putnam Mathematical Competition held on March 2, 1957, have been determined in accordance with the constitution of the competition. This competition is supported by the William Lowell Putnam Intercollegiate Memorial Fund left by Mrs. Putnam in memory of her husband and is held under the auspices of the Mathematical Association of America.

The first prize, four hundred dollars, is awarded to the Department of Mathematics of Harvard University, Cambridge, Massachusetts. The members of the team were Everett C. Dade, David B. Mumford, and Rohit J. Parikh; to each of these a prize of forty dollars is awarded.

The second prize, three hundred dollars, is awarded to the Department of Mathematics of Columbia College, New York, New York. The members of the team were David Bloom, Jonathan Lubin, and Jerrold Rubin; to each of these a prize of thirty dollars is awarded.

The third prize, two hundred dollars, is awarded to the Department of Mathematics of Cornell University, Ithaca, New York. The members of the team were Frans M. Djorup, Stanley Kaplan, and Robert F. Riley; to each of these a prize of twenty dollars is awarded.

The fourth prize, one hundred dollars, is awarded to the Department of Mathematics of the California Institute of Technology, Pasadena, California. The members of the team were Luis Baez-Duarte, H. Jerome Keisler, and Charles J. Stone; to each of these a prize of ten dollars is awarded.

The five persons ranking highest in the examination, named in alphabetical order, are David Bloom, Columbia College; Richard Bumby, Massachusetts Institute of Technology; Everett C. Dade, Harvard University; Rohit J. Parikh, Harvard University; and Ian Richards, University of Minnesota. Each of these will receive a prize of fifty dollars.

The six succeeding persons (tenth and eleventh tied) ranking highest in the examination, named in alphabetical order, are Armand Brumer, Brandeis University; H. Jerome Keisler, California Institute of Technology; David B. Mumford, Harvard University; Robert F. Riley, Cornell University; Paul A. Schweitzer, College of the Holy Cross; and Lawrence Shepp, Polytechnic Institute of Brooklyn. Each of these will receive a prize of twenty dollars.

The following teams, named in alphabetical order, won honorable mention: Oberlin College, Oberlin, Ohio, the members of the team being George Hannauer, Daniel Kleinman, and John Miller; Polytechnic Institute of Brooklyn, Brooklyn, New York, the members of the team being James Ax, Donald Passman, and Lawrence Shepp; University of California, Los Angeles, California, the members of the team being Richard S. Grote, Louis A. Jaeckel, and Michael Raugh; and University of Michigan, Ann Arbor, Michigan, the members of the

team being John S. Denton, Jr., James Penquite, and Charles Sims.

Twelve individuals were given honorable mention. The names, in alphabetical order, are: David M. Dahm, Rice Institute; John S. Denton, Jr., University of Michigan; John D. Ferguson, Yale University; Stanley Kaplan, Cornell University; Jacob Klein, College of the City of New York; Stephen Lichtenbaum, Harvard University; Jonathan Lubin, Columbia College; John P. McCabe, Manhattan College; Michael Raugh, University of California, Los Angeles; Robert Solovay, Harvard University; Harold Nathaniel Ward, Swarthmore College; and Peter Wolk, Massachusetts Institute of Technology.

A total of 505 individuals from 102 institutions entered the competition this year. Of this number 128 individuals and 9 institutions were unable to compete, due to various reasons. Therefore, a total of 377 undergraduates from 93 institutions actually took part in the competition.

The following is a list, in alphabetical order, of all colleges and univerities which entered teams in the Competition: Agricultural and Mechanical College of Texas, Agricultural and Technical College of North Carolina, Alabama Polytechnic Institute, Arizona State College (Flagstaff), Arizona State College (Tempe), Bethune-Cookman College, Brandeis University, Brooklyn College, Brown University, California Institute of Technology, Carleton College, Carnegie Institute of Technology, College of the Holy Cross, Columbia College, Cornell University, Dartmouth College, Davidson College, DePaul University, Georgia Institute of Technology, Harvard University, Haverford College, Kent State University, Kenyon College, Knox College, Lebanon Valley College, Manhattan College, Massachusetts Institute of Technology, McGill University, Muskingum College, New Mexico College of Agriculture and Mechanical Arts, New York University, Oberlin College, Phillips University, Polytechnic Institute of Brooklyn, Princeton University, Purdue University, Queen's University (Ontario), Rice Institute, Rose Polytechnic Institute, Royal Military College of Canada, Rutgers College of South Jersey, Saint Francis Xavier University, Saint Martin's College, Saint Mary's College (Indiana), Saint Olaf College, San Jose State College, Siena College, Spring Hill College, Stanford University, State College of Washington, Swarthmore College, The Cardinal Stritch College, The College of Saint Catherine, The College of the City of New York, The Ohio State University, The University of British Columbia, Union College, United States Naval Academy, University of Arizona, University of California (Berkeley), University of California (Davis), University of California (Los Angeles), University of Detroit, University of Michigan, University of Minnesota, University of Nebraska, University of Notre Dame, University of Oregon, University of Rochester, University of Tennessee, University of Toronto, University of Washington, Ursinus College, Vanderbilt University, Vassar College, Wayne State University, Wesleyan University (Connecticut), and Yale University.

The following colleges and universities, alphabetically arranged, entered individual contestants only: Blackburn College, Colorado School of Mines, Fordham University, Friends University, Hamilton College (of McMaster University), Iowa State College, Long Beach State College, Manchester College, Miami University, Northwestern University, Seattle University, Syracuse University, Talladega College, The College of Saint Francis, The College of Saint Thomas, The Cooper Union, The University of Texas, The University of Western Ontario, University of Alaska, University of Buffalo, University of Chicago, University of Cincinnati, Washington University (Missouri), and West Virginia Wesleyan.

The individual rankings of contestants (except for the relative ranks of the

first five) may be obtained by any department of mathematics for the purpose of selecting graduate students.

The problems given to those participating in the competition, together with a write-up of the solutions, will appear in a later issue of the MONTHLY.

MATHEMATICAL NOTES

EDITED BY IVAN NIVEN, University of Oregon

Material for this department should be sent to Ivan Niven, Department of Mathematics, University of Oregon, Eugene, Oregon.

THE NUMBER OF ELEMENTS OF GIVEN PERIOD IN FINITE SYMMETRIC GROUPS*

ROBERT B. HERRERA, Los Angeles City College and University of California, Los Angeles

1. Introduction. During the past ten years a number of papers on the solutions of $x^d=1$ in symmetric groups have appeared. In these papers asymptotic expansions for the number of elements whose period is a divisor of d have been obtained; the expansions in turn have been based on recursion formulas for the finite case. Jacobsthal [4] obtained a recursion formula and an asymptotic expansion for the case when d is a prime. Chowla and Herstein, together with Moore [1], and later with Scott [2], obtained properties of the recursion for d taken equal to 2, and then for d taken as any positive integer. The recursion formula for positive integral d, obtained by Chowla, Herstein and Scott, is equivalent to equation (3) given below.

More recently, general results in the asymptotic case have been obtained by Moser and Wyman [5], who have also extended the investigation of T_n, the number of solutions of $x^2=1$ in the symmetric group. By using Hermite polynomials and functions of a complex variable, Moser and Wyman have obtained refinements of expansions suggested in [2], together with further expansions for both the symmetric group and the alternating group.

The result obtained in the present paper is a complete expansion formula (in the finite case) for the number of elements of given period in finite symmetric groups. This result was first announced in 1946; it first appeared in print in 1953 [3].

2. Elements of period r in S_k. Let $S(k)$ be the set of all 1-1 mappings of the first k integers upon themselves, that is, permutations of $1, \cdots, k$, and let $S(k-1)$ be the subset of all mappings in $S(k)$ which leave k invariant, that is,

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* Presented to the Mathematics Seminar, University of California, Los Angeles, May 22, 1946, and, in revised form, to the Southern California Section of the Association, March 14, 1953.

The author wishes to thank Professor Leonard Carlitz for editorial advice in the preparation of this paper.

which map k onto k. Then $S(k)$ is a realization of the symmetric group of degree k and order $k!$, and $S(k-1)$ is the symmetric group of degree $k-1$ as a subgroup of $S(k)$.

Let $N(k; r)$ represent the number of elements in $S(k)$ of period r, with the convention that $N(0; 1)=1$. We shall prove the

THEOREM. $N(k; r)=\sum_s \{(k-1)!/(k-s)!\} \sum_b N(k-s; b)$, *where* $[s, b]=r$.

We require the

LEMMA. *If $S(k)$ be separated into cosets mod $S(k-1)$, such that $S(k)=S(k-1)+\sum_{i=1}^{k-1} X(i)$, where $X(i)$ is the right coset of $S(k-1)$ containing all the mappings in $S(k)$ which carry k into i, then each $X(i)$ contains the same number of elements of given period.*

Proof of the lemma. Let p be a permutation in $S(k-1)$ which carries i into j, then $p^{-1}X(i)p=X(j)$. Thus $X(i)$ and $X(j)$ are conjugate subsets of $S(k)$, and as such they contain the same number of elements of given period.

As a consequence of the lemma we have the

COROLLARY. $N(k; r)=N(k-1; r)+(k-1)N'(k; r)$, *where $N'(k; r)$ is the number of elements of period r in one of the cosets of $S(k-1)$.*

Proof of the theorem. Let $M(k; r)$ be the number of elements in $S(k)$ whose period is a divisor of r; let $S(k-1)$ be the subgroup of permutations in $S(k)$ leaving k invariant, and let $X(k-1)$ be the right coset of permutations in $S(k)$ mapping k upon $k-1$.

By the lemma we find

$$M(k; r) = M(k-1; r) + (k-1)M'(k; r), \tag{1}$$

where $M'(k; r)$ is the number of permutations in $X(k-1)$ whose period is a divisor of r.

The permutations from $X(k-1)$ have the form

$$(k, k-1, j_3, j_4, \cdots, j_s)\cdot p = c\cdot p,$$

where $s \mid r$, and p is a permutation of the integers other than $k, k-1, j_3, j_4, \cdots, j_s$ such that (period of p) $\mid r$.

The cycle c can be selected in exactly the following number of different ways, $(k-2)(k-3)\cdots(k-s+1)$, since j_3 is selected from among $k-2$ integers, j_4 from among $k-3, \cdots, j_s$ from among $k-(s-1)$ integers.

Having selected c, we may select permutation p in exactly $M(k-s; r)$ ways, as a permutation on $k-s$ symbols, whose period is a divisor of r. Thus we find

$$M(k; r) = M(k-1; r) + (k-1)\sum_{s \mid r} (k-2)(k-3)\cdots(k-s+1)M(k-s; r), \tag{2}$$

where $s \geqq 2$.

If the vacuous product is defined to be 1, then we have

$$M(k; r) = \sum_{s|r} (k-1)(k-2)\cdots(k-s+1)M(k-s; r),$$

which gives us

$$M(k; r) = \sum_{s|r} \frac{(k-1)!}{(k-s)!} M(k-s; r). \tag{3}$$

(Formula (3) is equivalent to formula (I), [2], the result obtained by Chowla, Herstein and Scott.)

To compute $N(k; r)$ we employ the number-theoretic combinatorial principle for selecting a defect-free subset from a given set of objects. Thus we find

$$N(k; r) = M(k; r) - \sum_{p|q} M(k; r/p) + \sum_{pq|r} M(k; r/pq) - \sum + \sum \cdots, \tag{4}$$

where $p, q, \cdots$, are distinct prime divisors of r.

The expansion (4) can be condensed by means of the Möbius function, $\mu(a)$, to the form

$$N(k; r) = \sum_{a|r} \mu(a) M(k; r/a). \tag{5}$$

Substituting for $M(k; r/a)$ in (5) from (3), we obtain

$$N(k; r) = \sum_{a|r} \mu(a) \left[\sum_{s|r/a} \frac{(k-1)!}{(k-s)!} M(k-s; r/a) \right]. \tag{6}$$

The sum in (6) can be expressed in the equivalent form

$$N(k; r) = \sum_{s|r} \frac{(k-1)!}{(k-s)!} \left\{ \sum_{b|r} \left[\sum_{c|b} \mu(c) M(k-s; b/c) \right] \right\}, \tag{7}$$

where $[b, s] = r$.

(Proof of the equivalence of (7) and (6) is quite detailed; the details are omitted in this paper.) Substituting from (5) into (7), we obtain the result announced in the theorem.

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A SIMPLER MATRIC APPROACH TO LINEAR DIFFERENTIAL EQUATIONS

M. F. SMILEY, State University of Iowa

In a recent paper, Barrett and Wylie [2] discuss systems of linear differential equations by using a bit of matric theory. It seems desirable to observe that matric theory has much more to offer to such a discussion than is brought out in [2]. The purpose of this note is to remind our readers that the Smith canonical form of a matrix with polynomial elements [1, p. 92] is just the tool one needs to provide a complete discussion of the systems considered in [2].

Consider the equation $AX=F$, where A is a *rectangular* matrix of polynomials in the differential operator D with constant coefficients, while X and F are vectors of unknown and known functions of t, respectively. According to Smith, there are nonsingular matrices P and Q whose elements are polynomials in D and such that PAQ has nonzero entries only on its diagonal. Then, clearly, $PAQQ^{-1}X=PF$ is a set of linear differential equations for the components of the unknown vector $Y=Q^{-1}X$, each of which involves just one component of Y. If we assume that these equations can be solved, then we can solve $AX=F$ by computing $X=QY$.

There are many texts (for example, [3, p. 82]) which tell us how to *compute* P and Q, and it is clear that we need not compute P when $F=0$. It should be emphasized that this procedure applies equally well if the entries of A happen to be independent of D. In this case the procedure is closely related to the familiar one for solving a system of linear algebraic equations.

The examples cited in [2] are, of course, very easily handled by the procedure of Smith. For the one on page 475, we find that

$$PAQ = \begin{bmatrix} 1 & 0 \\ 0 & f(D) \end{bmatrix} \qquad Q = \begin{bmatrix} 1 & g(D) \\ 0 & 1 \end{bmatrix}$$

with $f(x)=x^4+2x^3-8x+5$ and $g(x)=(1/8)(x^3-x^2-5x-3)$. If y_2 is the general solution of $f(D)y=0$, we obtain $x_1=g(D)y_2$, $x_2=y_2$ as the general solution of $AX=0$. In the example on page 478, we may take P and Q independent of D and reduce the example to the solution of

$$(D^2-4)y_1=(1/6)(2f_1+f_2), \qquad (D^2-1)y_2=f_1+2f_2.$$

We conclude with the observation that a complete discussion of systems of linear differential equations with constant coefficients cannot avoid the subject of invariant factors.

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ON CONTINUOUS DIFFERENTIABILITY

ROBERT WEINSTOCK, University of Notre Dame

It is well known that the derivative of a function differentiable at each point of an interval may exhibit discontinuities at points of the interval. An example frequently cited is the derivative of the everywhere continuous function defined, when $x \neq 0$, by $f(x) = x^2 \sin x^{-1}$; here $f'(x)$ exists for all x but is discontinuous at $x = 0$. Not so well known, however, is the simple relationship between differentiability and continuous differentiability which is the object of this note. The author has never seen the theorem in print, and he is quite surprised not to have found any one who claims to have previously known it—although several individuals have produced almost immediate proofs upon hearing a statement of it.

THEOREM. *Let the derivative $f'(x)$ exist on an open interval I, of which $a \leqq x \leqq b$ is a closed subinterval. A necessary and sufficient condition that $f'(x)$ be continuous on $a \leqq x \leqq b$ is the "uniform differentiability" of $f(x)$ on $a \leqq x \leqq b$—namely, that for each $\epsilon > 0$ there exist an x-independent $\delta > 0$ such that*

$$\left| \frac{f(x+h) - f(x)}{h} - f'(x) \right| < \epsilon \tag{1}$$

whenever $0 < |h| < \delta (a \leqq x \leqq b,\ x+h \in I)$.

Proof of sufficiency. By hypothesis, if $\{h_i\}$ is any sequence of nonzero numbers (with $x+h_i \in I$ for all x on $a \leqq x \leqq b$) that converges to zero, then the sequence of *continuous* functions defined by $\{[f(x+h_i) - f(x)]/h_i\}$ converges *uniformly* to $f'(x)$ on $a \leqq x \leqq b$. The limit $f'(x)$ is therefore continuous on $a \leqq x \leqq b$.

Proof of necessity. The mean-value theorem provides:

$$\frac{f(x+h) - f(x)}{h} - f'(x) = f'(x+\theta h) - f'(x) \qquad (0 < \theta < 1) \tag{2}$$

for each x on $a \leqq x \leqq b$ and $x+h \in I$. Since $f'(x)$ is by hypothesis continuous, and therefore uniformly continuous, on $a \leqq x \leqq b$, there exists for each $\epsilon > 0$ an x-independent $\delta > 0$ such that

$$|f'(x+\theta h) - f'(x)| < \epsilon \tag{3}$$

whenever $0 < |\theta h| < \delta$ and, therefore, whenever $|h| < \delta$. The required uniformity of differentiability (1) follows from (2) and (3).

ON TAYLOR'S THEOREM WITH REMAINDER

P. H. DIANANDA, University of Malaya

1. Introduction and summary. In this note I prove Taylor's theorem, with the Schlömilch-Roche form of the remainder, in the following form:

THEOREM A. *If*
(1) $f(x)$ *is continuous in* $a \leqq x \leqq b (=a+h)$,
(2) $f^{(n-1)}(x)$ *is continuous in* $a \leqq x < b$,
(3) $f^{(n)}(x)$ *exists in* $a < x < b$,
then, for each $\nu < n$,

$$f(b) = f(a) + hf'(a) + (h^2/2!)f''(a) + \cdots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a) + \frac{h^n(1-\theta)^\nu}{(n-1)!(n-\nu)} f^{(n)}(a+\theta h),$$

for some θ *such that* $0 < \theta < 1$.

Ordinarily this theorem is proved under the more restrictive assumptions obtained by replacing the interval $a \leqq x < b$ in (2) by the closed interval $a \leqq x \leqq b$. Stolz (see [1]) proved the theorem in the form stated above, but, as W. H. Young [3] and Hobson [1] have noted, Stolz's proof breaks down whenever ν is a positive integer. His proof made use of Cauchy's formula. The present proof, based on the usual proof of the theorem in its ordinary form, makes use of an extension of Rolle's theorem and one of l'Hospital's rules.

2. Preliminaries. The following forms of Rolle's theorem (see [1]) and of l'Hospital's rule [2] are used in proving Theorem A.

THEOREM 1. *If*
(1) $F(x)$ *is continuous in* $a \leqq x < b$,
(2) $F(b-0)$ *either does not exist uniquely or exists uniquely and equals* $F(a)$,
(3) $F'(x)$ *exists in* $a < x < b$,
then $F'(\xi) = 0$ *for some* ξ *such that* $a < \xi < b$.

THEOREM 2. *If*
(1) $g(x)$, $G(x)$ *are continuous in* $a < x < b$,
(2) $g'(x)$, $G'(x)$ *exist finitely in* $a < x < b$,
(3) $G(b-0)$ *exists uniquely and is infinite*,
(4) $g'(x)/G'(x) \to l$ *(finite or infinite) as* $x \to b-0$,
then $g(x)/G(x) \to l$ *as* $x \to b-0$.

3. Proof of Theorem A. For $a \leqq x < b$, let

$$F(x) = f(b) - f(x) - (b-x)f'(x) - \frac{(b-x)^2}{2!} f''(x) - \cdots - \frac{(b-x)^{n-1}}{(n-1)!} f^{(n-1)}(x) - \frac{(b-x)^{n-\nu}}{(n-1)!(n-\nu)} K,$$

where K is a constant so chosen as to make $F(a) = 0$. Now $F(b-0)$ either exists uniquely or does not so exist. The conditions of Theorem 1 are therefore satisfied if the following lemma holds.

LEMMA 1. *If $F(b-0)$ exists uniquely, then $F(b-0)=0$.*

Assuming the truth of this lemma we can now apply Theorem 1 to obtain that $F'(a+\theta h)=0$ for some θ such that $0<\theta<1$. Since, for $a<x<b$,

$$F'(x) = -\frac{(b-x)^{n-1}}{(n-1)!}f^{(n)}(x) + \frac{(b-x)^{n-r-1}}{(n-1)!}K,$$

Theorem A then follows.

Remark. $F(b)=0$ under the conditions ordinarily assumed in the proof of Taylor's theorem. Complications arise in our proof since nothing is known about $f'(b-0), f''(b-0), \cdots$.

We now proceed to the proof of Lemma 1. This we prove by repeated application of

LEMMA 2. *For $k=0, \cdots, n-1$, let*

$$\phi_k(x) \equiv a_{0,k}f(x) + a_{1,k}(b-x)f'(x) + \cdots + a_{k,k}(b-x)^k f^{(k)}(x).$$

If, for some $r(0<r<n)$ and for some finite constants $a_{0,r}, \cdots, a_{r,r}$, $\phi_r(b-0)$ exists uniquely (finite or infinite) and $\phi_r(b-0)\neq a_{0,r}f(b)$, then finite constants $a_{0,r-1}, \cdots, a_{r-1,r-1}$ exist such that $\phi_{r-1}(b-0)$ exists uniquely and is infinite.

Proof. It can be verified that $(b-x)\phi'_{r-1}(x)=\phi_r(x)-a_{0,r}f(x)$ if

$$\begin{aligned} a_{r-1,r-1} &= a_{r,r}, \\ a_{r-2,r-1} &= a_{r-1,r} + (r-1)a_{r-1,r-1}, \\ &\cdots\cdots\cdots\cdots\cdots \\ a_{0,r-1} &= a_{1,r} + a_{1,r-1}. \end{aligned}$$

These equations define finite constants $a_{r-1,r-1}, \cdots, a_{0,r-1}$ uniquely in terms of $a_{r,r}, \cdots, a_{1,r}$. With $\phi_{r-1}(x)$ determined as above we now set $g(x)=\phi_{r-1}(x)$, $G(x)=-\log(b-x)$, $l=\phi_r(b-0)-a_{0,r}f(b)$. Then, noting that $g'(x)/G'(x)=(b-x)\phi_{r-1}(x)=\phi_r(x)-a_{0,r}f(x)\to l$ as $x\to b-0$, we obtain, by Theorem 2, that

$$\frac{g(x)}{G(x)} = \frac{\phi_{r-1}(x)}{-\log(b-x)} \to l \qquad (x \to b-0).$$

Since $l\neq 0$ and $-\log(b-x)\to\infty$ as $x\to b-0$, it follows from the above that $\phi_{r-1}(b-0)$ exists uniquely and is infinite. We thus have Lemma 2.

To prove Lemma 1 we assume that $F(b-0)$, which exists uniquely, is not zero and obtain a contradiction. Our assumption gives us that $\phi_{n-1}(b-0)=f(b)-F(b-0)\neq f(b)$, on setting $a_{s,n-1}=1/s!$ $(s=0, \cdots, n-1)$. Repeated application of Lemma 2 (with $r=n-1$, $r=n-2, \cdots, r=1$) now gives us that the functions $\phi_{n-2}(x), \phi_{n-3}(x), \cdots, \phi_0(x)$ defined by Lemma 2 are such that each of them has a unique infinite limit as $x\to b-0$.

But now, $\phi_0(x)=a_{0,0}f(x)$, so that $\phi_0(b-0)=a_{0,0}f(b)$, which is finite. We thus

have a contradiction and this shows that $F(b-0)$, if it exists uniquely, is zero. This completes the proof of Lemma 1 and thus of Theorem A.

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A DETERMINANT INEQUALITY FOR UNIVALENT FUNCTIONS

S. D. BERNARDI, New York University

Let $w=f(z)=z+a_2z^2+a_3z^3+\cdots\in(S)$ denote the class of functions regular and univalent for $|z|<1$. Form

$$F(z,\alpha)=[z^{-1}f(z)]^{-\alpha/2}=1+b_1(\alpha)z+b_2(\alpha)z^2+\cdots, \tag{1}$$

having the coefficient relation

$$nb_n(\alpha)+\sum_{i=1}^{n}(\alpha i/2+n-i)a_{i+1}b_{n-i}(\alpha)=0; \qquad n=1,2,\cdots, b_0(\alpha)=1. \tag{1a}$$

Then, by Prawitz [1, p. 273] it follows that

$$\sum_{n=1}^{\infty}\{(2n-\alpha)/\alpha\}\,|\,b_n(\alpha)\,|^2\leqq 1, \text{ for all real } \alpha. \tag{2}$$

The inequality (2) which is a generalization of the so-called "area principle" has been used [1, p. 268] a great deal in the classical analysis of the theory of univalent (schlicht) functions. Numerical upper bounds for certain coefficients $b_n(\alpha)$ for special values of α have been estimated in order to find upper bounds for certain coefficients [1, p. 274] a_n of $f(z)$. Our results are summarized in the following theorem.

THEOREM. *Let $f(z)\in(S)$. Let $F(z,\alpha)$ be defined as in* (1). *Let*

$$z=f^{-1}(w)=w+\sum_{n=2}^{\infty}(c_{n-1}/n)w^n \tag{3}$$

where $f^{-1}(w)$ denotes the function inverse to f. Then

(a) $c_{n-1}=b_{n-1}(2n)$, $\qquad n=2,3,\cdots$

(b) $\left|\dfrac{b_{n-1}(2n)}{n}\right|\leqq H(n)=\dfrac{1\cdot3\cdot5\cdots(2n-1)}{1\cdot2\cdot3\cdots(n+1)}2^n$, $\qquad n=2,3,\cdots$

(c) $\displaystyle\sum_{n=3}^{\infty}(n-2)\,|\,b_n(4)\,|^2\leqq 18.$

Proof. Applying Lagrange's formula [3, p. 125] for the reversion of series, we have

$$z = f^{-1}(w) = \sum_{n=1}^{\infty} \frac{w^n}{n!}\left[\frac{d^{n-1}}{dz^{n-1}}\left(\frac{z}{f(z)}\right)^n\right]_{z=0}.$$

A formal computation yields

$$\left[\frac{d^{n-1}}{dz^{n-1}}\left(\frac{z}{f(z)}\right)^n\right]_{z=0} = (n-1)!\,b_{n-1}(2n), \qquad n = 1, 2, \cdots; b_0(2) = 1.$$

We thus obtain

$$z = f^{-1}(w) = \sum_{n=1}^{\infty} (1/n)b_{n-1}(2n)w^n. \tag{4}$$

Comparing (3) and (4) we get

$$c_{n-1} = b_{n-1}(2n); \qquad n = 1, 2, \cdots. \tag{5}$$

It has been shown by Löwner [1, p. 286] that the coefficients of the inverse function $z=f^{-1}(w)$ of a univalent function $f(z)$ are majorized by the coefficients of the inverse function $K^{-1}(w)$ of the Koebe function $w=K(z)=z/(1-z)^2$. Since

$$\begin{aligned} z = K^{-1}(w) &= w + \sum_{n=2}^{\infty} (-1)^{n-1} \frac{(2n)(2n-1)(2n-2)\cdots(n+2)}{n!} w^n \\ &= w + \sum_{n=2}^{\infty} (-1)^{n-1} H(n), \end{aligned} \tag{6}$$

it follows from (3) and (6) that $|c_{n-1}/n| \leqq H(n)$. Substituting from (5) we obtain (b) of the theorem. Letting $\alpha=4$ in (2) and using $|b_1(4)| = |-2a_2| \leqq 2H(2)=4$ yields (c) of the theorem, and also shows that the inequality (b) is sharp since $|a_2| \leqq 2$ is a well known sharp result. This completes the proof of the theorem.

If we let $\alpha=2k$, the coefficient relation in equation (1a) becomes $nb_n(\alpha) + \sum_{i=1}^{n} (ik+n-i)a_{i+1}b_{n-i}(\alpha)=0$; $n=1, 2, \cdots, b_0(\alpha)=1$. In determinant notation this is equivalent to

$$\frac{k}{n!}\begin{vmatrix} a_2 & 1 & 0 & 0 & \cdots & 0 \\ 2a_3 & (k+1)a_2 & 2 & 0 & \cdots & 0 \\ 3a_4 & (2k+1)a_3 & (k+2)a_2 & 3 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ na_{n+1} & ((n-1)k+1)a_n & ((n-2)k+2)a_{n-1} & ((n-3)k+3)a_{n-2} & \cdots & (k+n-1)a_2 \end{vmatrix}$$
$$= (-1)^n b_n(2k) = \Delta.$$

Thus, the inequality relation (b) of the theorem offers certain upper bounds for the determinant Δ. The special case of the determinant Δ obtained by taking

$k=1$ and restricting the coefficients a_n to be rational integers is considered in [4]. Another special case of Δ obtained by letting $k=1$ and restricting the coefficients a_n to be integers in an imaginary quadratic field is considered in [2]. It seems reasonable to suppose that the inequality (b) acting upon Δ plus the imposition of special properties on the coefficients a_n should be a source of many special results.

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CLASSROOM NOTES

EDITED BY C. O. OAKLEY, Haverford College

All material for this department should be sent to C. O. Oakley, Department of Mathematics, Haverford College, Haverford, Pa.

TRANSFORMATION OF COORDINATES

C. S. OGILVY, Hamilton College

In a note, *Finding the cartesian equation of a locus*, this MONTHLY, vol. 63, pp. 661–663, the cartesian equation of $r=\sin(\theta/3)$ is found by a new method involving a certain amount of laborious reduction. The remark is made that the "example would be extremely difficult to carry out without the aid of a special device." Actually the cartesian equation can be found by the following simple procedure.

In the standard formula $\sin 3\phi=3\sin\phi-4\sin^3\phi$, write $\phi=\theta/3$. Then $\sin\phi=\sin(\theta/3)=r$, yielding $\sin\theta=3r-4r^3$. Multiplying through by r, $y=3(x^2+y^2)-4(x^2+y^2)^2$.

A METHOD FOR SOLVING $\phi(x)=n$

L. L. PENNISI, University of Illinois, Chicago

This note reports a method for finding all the values of x, when given an integer n, such that

$$\phi(x) = n, \tag{1}$$

where $\phi(x)$ is the Euler ϕ-function of x.

Let

$$d_1, d_2, \cdots \tag{2}$$

be all the divisors of n satisfying the condition that neither d_i nor n/d_i is equal to $2k+1$, $k>0$.

For each d_i, let

$$E_{i,1}, E_{i,2}, \cdots \tag{3}$$

denote all the solutions of the equation $\phi(y)=d_i$ having the form p^e, where p is a prime number.

Remark. Such solutions need not necessarily exist. For example, $\phi(y)=50$ has no solutions of the form p^e. Also $E_{i,j}/d_i=p^e/p^{e-1}(p-1)=p/(p-1)\leqq 2$. Therefore, $E_{i,j}\leqq 2d_i$, that is, the solutions of $\phi(y)=d_i$ of the form p^e cannot be greater than $2d_i$.

Let

$$n = d_{i_1}\cdot d_{i_2}\cdot\cdots$$

be a factorization of n whose factors satisfy the condition (2). Also, let

$$E_{i_1,j_1}, E_{i_2,j_2}, \cdots$$

be relatively prime, where the E's are as in (3).

Then

$$x = E_{i_1,j_1}\cdot E_{i_2,j_2}\cdot\cdots \tag{4}$$

is a solution of equation (1).

For, since the E's are relatively prime, we have

$$\phi(x) = \phi(E_{i_1,j_1})\cdot\phi(E_{i_2,j_2})\cdot\cdots = d_{i_1}\cdot d_{i_2}\cdot\cdots = n.$$

Every solution of equation (1) *has the form given in* (4). For, suppose that x_0 is a solution of $\phi(x)=n$. Then, factoring x_0 into prime powers we get $x_0=p_1^{e_1}\ p_2^{e_2}\cdot\cdots$. Hence $n=\phi(x_0)=\phi(p_1^{e_1})\cdot\phi(p_2^{e_2})\cdot\cdots$. Consequently, $\phi(p_1^{e_1})$, $\phi(p_2^{e_2})$, $\cdots$, are divisors of n and neither $\phi(p_i^{e_i})$ nor $n/\phi(p_i^{e_i})$ has the form $2k+1$, $k>0$. Hence, $\phi(p_1^{e_1})=d_{i_1}$, $\phi(p_2^{e_2})=d_{i_2}$, $\cdots$, where the d's are as described in (2). Therefore, $p_1^{e_1}=E_{i_1,j_1}$, $p_2^{e_2}=E_{i_2,j_2}$, $\cdots$, where the E's are as described in (3), and are relatively prime. Therefore $x_0=E_{i_1,j_1}\cdot E_{i_2,j_2}\cdot\cdots$.

EXAMPLE. Solve $\phi(x)=12$.

Using (2), we find $d_1=1$, $d_2=2$, $d_3=6$, $d_4=12$. Using (3), we have:

$\phi(y)=d_1=1$ implies that $y=1$, $y=2$. Take $E_{1,1}=1$, $E_{1,2}=2$.

$\phi(y)=d_2=2$ implies that $y=3$, $y=2^2$. Take $E_{2,1}=3$, $E_{2,2}=4$.

$\phi(y)=d_3=6$ implies that $y=7$, $y=3^2$. Take $E_{3,1}=7$, $E_{3,2}=9$.

$\phi(y)=d_4=12$ implies that $y=13$. Take $E_{4,1}=13$.

Using (4), we find all the values of x to be:

$$x = E_{1,1} \cdot E_{4,1} = 1 \cdot 13 = 13.$$
$$x = E_{1,2} \cdot E_{4,1} = 2 \cdot 13 = 26.$$
$$x = E_{2,1} \cdot E_{3,1} = 3 \cdot 7 = 21.$$
$$x = E_{2,2} \cdot E_{3,1} = 4 \cdot 7 = 28.$$
$$x = E_{2,2} \cdot E_{3,2} = 4 \cdot 9 = 36.$$
$$x = E_{1,2} \cdot E_{2,1} \cdot E_{3,1} = 2 \cdot 3 \cdot 7 = 42.$$

A NEW PROOF THAT NO PERMUTATION IS BOTH EVEN AND ODD

J. L. BRENNER, Stanford Research Institute

This note contains an informal proof that no permutation (on a finite set) is both even and odd. The parallel formal (abstract) proof would begin with postulates which "permutation" must satisfy, and would derive the same conclusion from these postulates. We know one other informal proof which can be made formal in this sense [1, p. 36]; the postulates which would be needed to make that proof abstract seem to us to be less simple. A third proof [3, p. 8] is very short, but not abstract; in this third proof, the objects on which the permutations act play an essential role in the argument. In the first two proofs this is not the case; the objects on which the permutations act do not even appear in the formal versions of the proofs.

To carry through the proof, we need to use the following facts: (1) Every permutation can be written as a product of transpositions; (2) mutually disjoint transpositions commute; (3) a transposition is its own inverse; (4) if a_r, a_s are distinct, then $(a_s, a_t)(a_r, a_t) = (a_r, a_t)(a_s, a_r)$; (5) if all the symbols $a_t, a_{t+1}, \cdots, a_{r-1}, a_r$ $(r > t)$ are distinct, then the first and last transpositions in the product

$$(a_t, a_{t+1})(a_{t+1}, a_{t+2}) \cdots (a_{r-1}, a_r)(a_r, a_t)e$$

can be omitted. Now the informal proof.

No permutation on a finite set can be expressed as a product of an even, and also as a product of an odd number of transpositions. Thus, by (1), *every permutation can be classified as either even or odd.*

Proof. Assume that the permutation P can be written in two ways, first as a product of an even number, and then as a product of an odd number of transpositions. We equate these products, use (3), and have a representation of the identity e as a product of an odd number $2k+1$ of transpositions. We show that if such a representation exists, there must be another such representation in which the number of factors is two fewer; this is clearly a contradiction,* since e is not equal to a single transposition.

Of all the ways in which the identity can be written as a product of $2k+1$

* This is a contradiction since if there were a representation, there would be one in which the number of factors is a minimum.

transpositions, let us take a representation in which the "initial chain" $\{\ \}$ is as long as possible:

$$\epsilon = \{(a_1a_2)(a_2a_3)\cdots(a_{r-1}a_r)\}\cdots,$$

where $a_1, a_2, \cdots, a_r$ are all distinct. Clearly $r>1$, and one or more transpositions (a_r, b) must follow this initial chain $(b\neq a_r)$. Without changing the total number of factors, or the number of factors in the initial chain, let us suppose that such an additional factor follows the initial chain as closely as possible. By (4), the additional factor must follow the initial chain immediately:

$$\epsilon = \{(a_1a_2)(a_2a_3)\cdots(a_{r-1}a_r)\}(a_rb)\cdots.$$

But this completes the argument. For b is not distinct from $a_1, \cdots, a_r$ since the chain is already as long as possible. Hence $b=a_t$ for some t, $1\leqq t<r$, and we use (5).

It is known that the permutations on an infinite set cannot be classified as even or odd [2].

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REDUCING THE EQUATION OF THE GENERAL CONIC

R. Lariviere, University of Illinois, Chicago

Mathematicians, hoping to introduce many new topics into elementary courses, are scrutinizing the content of the latter and criticizing not only the material comprised but also the manner in which some of it is submitted to the students. This is causing new types of presentation, intended to save time and promote understanding, to become increasingly prevalent in recently published or recently revised textbooks.

Rotation of rectangular coordinate axes in two dimensional space to simplify the equation of the general conic is a topic that can profit from a change of procedure. In practice the treatment of rotation has frequently been curtailed in the simpler textbooks because examples applying the process involved so much computation that classes rarely had time to develop any mastery of the required techniques before having to abandon rotation for other work.

1. It is now possible to unify the treatment of all types of conics and to reduce the computational difficulty materially without eliminating the needed drill in the direct application of the equations of rotation. To take advantage of these procedures an author merely needs to add the following theorem to the discussion in his text.

Theorem I. *If the rectangular coordinate axes referred to in the equation*

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0, \qquad B \neq 0, \tag{1}$$

are rotated to eliminate the xy term and to produce

$$A'X^2 + C'Y^2 + D'X + E'Y + F = 0, \tag{2}$$

then A', the coefficient of X^2, is equal to $[A+(B/2)\tan\theta]$ and C', the coefficient of Y^2, is equal to $[C-(B/2)\tan\theta]$, where θ is the angle of rotation [1].

Since, in rotation, the quadratic terms are independent of the linear and *vice versa*, equation (2) may be derived from

$$[A + (B/2)\tan\theta]X^2 + [C - (B/2)\tan\theta]Y^2 + D[X\cos\theta - Y\sin\theta] + E[X\sin\theta + Y\cos\theta] + F = 0.$$

The student then completes the square(s) to find the center or the vertex and the reduced equation is apparent.

Theorem I above can readily be proved by substituting the equations of rotation in (1) to obtain the values of A', C', and B'. A condition of the theorem is that $B'=0$. We then have two cases:

Case I. $A \neq C$. $A'+C'=A+C$, $\tan 2\theta = B/(A-C)$, $A'-C'=B/\sin 2\theta$. Hence $2A' = A + C + B/\sin 2\theta = 2A - B/\tan 2\theta + B/\sin 2\theta$. Therefore $A' = A + (B/2)\tan\theta$. Similarly $C' = C-(B/2)\tan\theta$.

Case II. $A=C$. $B'=B\cos 2\theta=0$, $\cos 2\theta=0$, $2\theta=(2n-1)\pi/2$. $\tan\theta\sin 2\theta>0$, $\tan\theta=\sin 2\theta$. $A'=A\cos^2\theta+B\sin\theta\cos\theta+C\sin^2\theta=A+(B/2)\sin 2\theta=A+(B/2)\tan\theta$. In like manner C' may be shown to be $C-(B/2)\tan\theta$.

This concludes the proof of the theorem. An alternative proof can be obtained by extending the algebraic derivation of the sine and cosine of the angle of rotation in [2].

2. The following is another useful result [3].

THEOREM II.

$$\operatorname{Tan}\theta = \frac{C-A}{B} \pm \sqrt{\left(\frac{C-A}{B}\right)^2 + 1}.$$

Instead of deriving a value of θ from $\tan 2\theta$ when $A \neq C$ or $\cos 2\theta$ when $A=C$, we may, since $\cos\theta\neq 0$, obtain θ by dividing the equation $B'=0$ by $-2B\cos^2\theta$ and solving directly for $\tan\theta$.

If the least rotation is required we again have two cases:

Case I. $A\neq C$. The sign chosen to precede the radical must be opposite that of $(C-A)/B$, and $90°>\theta>-90°$.

Case II. $A=C$. $\operatorname{Tan}\theta=\pm 1$, $\theta=(2n-1)\pi/4$, and the least rotations are $\pm\pi/4$.

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MULTIPLICATION TABLES FOR TRIGONOMETRIC OPERATORS

G. A. BAKER, University of California, Davis

The lists of exercises given in some books on trigonometry imply that the operations of taking direct and inverse trigonometric functions can be done in either order. It is the purpose of this note to examine the validity of such an implication and to give multiplication tables for trigonometric operators for suitably restricted values. It is found that, except for the diagonal elements, different results are obtained depending on the order in which the operators are applied. That is, the operators are not commutative. Some combinations are not possible, many others are not distinct, and still others exhibit interesting patterns of symmetry.

Consider the following multiplication table where the multiplication is taken in the order (top symbol) $\times$ (side symbol). For simplicity the angles are restricted to be between 0 and $\pi/2$.

	Sin	Cos	Tan	Cot	Csc	Sec
Sin^{-1}	x	$\sqrt{1-x^2}$	$x/\sqrt{1-x^2}$	$\sqrt{1-x^2}/x$	$1/x$	$1/\sqrt{1-x^2}$
Cos^{-1}	$\sqrt{1-x^2}$	x	$\sqrt{1-x^2}/x$	$x/\sqrt{1-x^2}$	$1/\sqrt{1-x^2}$	$1/x$
Tan^{-1}	$x/\sqrt{1+x^2}$	$1/\sqrt{1+x^2}$	x	$1/x$	$\sqrt{1+x^2}/x$	$\sqrt{1+x^2}$
Cot^{-1}	$1/\sqrt{1+x^2}$	$x/\sqrt{1+x^2}$	$1/x$	x	$\sqrt{1+x^2}$	$\sqrt{1+x^2}/x$
Csc^{-1}	$1/x$	$\sqrt{x^2-1}/x$	$1/\sqrt{x^2-1}$	$\sqrt{x^2-1}$	x	$x/\sqrt{x^2-1}$
Sec^{-1}	$\sqrt{x^2-1}/x$	$1/x$	$\sqrt{x^2-1}$	$1/\sqrt{x^2-1}$	$x/\sqrt{x^2-1}$	x

It is understood that Sin^{-1}, for instance, is applied to a suitable quantity x so that $\text{Sin} \times \text{Sin}^{-1}$ means $\text{Sin}\ (\text{Sin}^{-1}\ x)$, *etc.* The range of x in each case is obvious from the heading.

Now consider the similar multiplication table where the multiplication is taken in the order (side symbol) $\times$ (top symbol).

The symbol $\text{Sin}^{-1}\ (\text{Csc}\ x)$ means the angle whose sine is $\text{Csc}\ x$. Since the range of $|\text{Csc}\ x|$ is from 1 to infinity, there is just one value $0 \leqq x \leqq \pi/2$, namely $\pi/2$, for which $\text{Sin}^{-1}\ (\text{Csc}\ x)$ exists. This fact is indicated by a bar. Similar situations obtain for the other spaces designated by bars in the second table.

The symbol $\text{Sin}^{-1}\ (\text{Tan}\ x)$ means the angle whose sine is $\text{Tan}\ x$. Since the sine can not be greater than one in absolute value, x must lie between 0 and $\pi/4$ if we consider angles in the first quadrant. For a given x we could find the corresponding value of $\text{Sin}^{-1}\ (\text{Tan}\ x)$ by means of a table by first finding $\text{Tan}\ x$

	Sin	Cos	Tan	Cot	Csc	Sec
Sin^{-1}	x	$\frac{\pi}{2}-x$	*	*	—	—
Cos^{-1}	$\frac{\pi}{2}-x$	x	*	*	—	—
Tan^{-1}	*	*	x	$\frac{\pi}{2}-x$	*	*
Cot^{-1}	*	*	$\frac{\pi}{2}-x$	x	*	*
Csc^{-1}	—	—	*	*	x	$\frac{\pi}{2}-x$
Sec^{-1}	—	—	*	*	$\frac{\pi}{2}-x$	x

and then looking up the value of the angle whose sine is the given value of Tan x. Geometrically we could construct the unit circle; draw the tangent at the point $(1, 0)$; draw the angle x and let the terminal side intersect the tangent; let the line $y = \text{Tan } x$ intersect the circle and connect the point with the origin. This line is the terminal side of the angle Sin^{-1} (Tan x). The situations indicated by asterisks in the second table are similar. In these cases, functions of x exist over parts of the interval $0 \leqq x \leqq \pi/2$.

SPEAK UP

The excellent *Suggestions to students on talking about mathematics*, this MONTHLY, vol. 64, 1957, pp. 16–18, omits one of the most important suggestions that should be made: SPEAK UP!

So many speakers, notable men whose talks have been eagerly expected, fail to deliver, because they speak in a conversational tone, drop their voices on points of especial interest, and talk to the blackboard instead of to their hearers.

I once saw a distinguished man deliberately turn away from the microphone furnished and let his moderate voice die out over the first few rows.

Another began by saying he would divide his subject into two parts; he drew a vertical line on the blackboard, added nothing more, but kept his eye upon it and talked to it throughout his lecture.

These bad habits seem to be on the increase among learned people. At recent scientific meetings that I have attended, not one speaker in five has taken enough trouble to make his material audible to his whole audience.

W. R. RANSOM
Tufts University

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1275 [1957, 432]. *Corrected*

Solve for x:

$$\int_0^x s^{8/3}(1-s)^{4/3}ds = \int_0^1 t^{8/3}(1+t)^{-6}dt.$$

E 1276. *Proposed by C. S. Ogilvy, Hamilton College*

Find the greatest right circular cylinder coaxal with and inscribed in the solid formed by rotating around the y-axis the area bounded by the two axes, the parabola $y=9x^2-28x+24$, and the parabola's minimum ordinate.

E 1277. *Proposed by Alexander Oppenheim, University of Malaya, Singapore, Malaya*

For each $p>0$ there is a greatest q and a least r such that

$$\frac{q \sin x}{1+p\cos x} \leqq x \leqq \frac{r \sin x}{1+p\cos x}$$

for $0 \leqq x \leqq \pi/2$. Determine q and r as functions of p.

E 1278. *Proposed by V. F. Ivanoff, San Carlos, Calif.*

If a hexagon $ABCDEF$ is inscribed in a conic, and if P is any point either on the conic or on the Pascal line of the hexagon, then

$$(\sin APB \sin CPD \sin EPF)/(\sin BPC \sin DPE \sin FPA) = \text{constant}.$$

E 1279. *Proposed by L. R. Ford, Jr., and D. R. Fulkerson, The RAND Corporation, Santa Monica, Calif.*

In the February 1957 issue of *Scientific American* the following problem appeared: "A carpenter, working with a buzz saw, wishes to cut a wooden cube, three inches on a side, into 27 one-inch cubes. He can do this easily by making six cuts through the cube, keeping the pieces together in the cube shape. Can he reduce the number of necessary cuts by rearranging the pieces after each cut?" Generalize this to find the number of cuts necessary to dissect an $n\times n\times n$ cube into n^3 one-inch cubes.

E 1280. *Proposed by V. K. Narayanan, Puthenchanthai, Trivandrum, South India*

If $x=24$, then $x+1$ and $2x+1$ are perfect squares; if $x=40$, then $2x+1$ and $3x+1$ are perfect squares; if $x=8$, then $x+1$ and $3x+1$ are perfect squares. Is there a positive integer x such that all three $x+1$, $2x+1$, $3x+1$ are perfect squares?

SOLUTIONS

Cones and Trihedral Angles

E 1246 [1957, 42]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Determine the relation between the radius of the base and the altitude of a right circular cone in which a trihedral angle can be inscribed whose face angles are all equal to a given angle 2α.

Show that if two trihedral angles whose face angles are all equal to 2α and $2\alpha'$, respectively, are inscribed in two right circular cones having a common base, then a necessary and sufficient condition for the radius of the common base to be a mean proportional between the altitudes of the cone is that

$$\sin^2 \alpha + \sin^2 \alpha' = 3/4.$$

Solution by C. F. Pinzka, Xavier University, Cincinnati, Ohio. Let r and h denote the radius of the base and altitude, respectively, of the right circular cone. The specified trihedral angle and base of the cone form a regular pyramid whose lateral edge is $(r^2+h^2)^{1/2}$ and whose base has sides of $\sqrt{3}r/2$. The bisector of a face angle forms a right triangle in which

$$\sin^2 \alpha = 3r^2/4(r^2 + h^2).$$

the required relationship.

If $\sin^2 \alpha+\sin^2 \alpha'=3/4$, then

$$r^2/(r^2 + h^2) + r^2/(r^2 + h'^2) = 1,$$

which leads easily to $r=(hh')^{1/2}$. The converse follows just as readily.

Also solved by Leon Bankoff, A. L. Epstein, Hazel Evans, P. L. Falb, Michael Goldberg, Cornelius Groenewoud, Vern Hoggatt, A. R. Hyde, I. M. Isaacs, J. D. E. Konhauser, Andrew Kraus, Josef Langr, D. C. B. Marsh, Beckham Martin, C. S. Ogilvy, D. S. Passman, P. A. Penzo, P. W. A. Raine, L. A. Ringenberg, D. A. Robinson, Azriel Rosenfeld, Chih-yi Wang, R. H. Wilson, Jr., and the proposer.

The relationship of the first part of the problem may be put into the form

$$h/r = [(3/4) \csc^2 \alpha - 1]^{1/2}.$$

An instance of the second part of the problem is $\alpha=45°$ and $\alpha'=30°$.

An Identity

E 1247 [1957, 42]. *Proposed by C. W. Topp, Fenn College*
Prove that

$$\sum_{k=1}^{n} (-1)^{k+1}\binom{n}{k}\binom{kn}{n} = (-1)^{n+1}n^n.$$

Solution by J. E. Darraugh and F. D. Parker, Clarkson College. Since

$$a^n - b^n = (a-b)\sum_{i=0}^{n-1} a^i b^{n-1-i},$$

we have

$$[1-(x+1)^n]^n = (-x)^n\left[\sum_{i=0}^{n-1}(x+1)^{n-1-i}\right]^n.$$

But

$$[1-(x+1)^n]^n = \sum_{i=0}^{n}(-1)^i\binom{n}{i}(x+1)^{ni}.$$

Since the coefficients of x^n in the two expansions are equal, we obtain

$$(-1)^n n^n = \sum_{k=1}^{n}(-1)^k\binom{n}{k}\binom{kn}{n},$$

from which the desired result immediately follows.

Also solved by W. A. Al-Salam, Leonard Carlitz, C. C. Conley, M. P. Drazin, N. J. Fine, Virginia S. Hanly, J. H. Hodges, A. R. Hyde, P. G. Kirmser, M. S. Klamkin, D. C. B. Marsh, T. F. Mulcrone, L. A. Ringenberg, D. A. Robinson, Chih-yi Wang, David Zeitlin, and the proposer. Late solution by J. L. Brown, Jr.

Several solvers obtained the desired result by setting $a=0$ and $b=n$ in the more general identity

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{a+bk}{n} = (-1)^n b^n, \qquad n \geqq 0.$$

This identity may be found, established by finite difference methods, in H. W. Gould, *Some generalizations of Vandermonde's convolution*, this MONTHLY [1956, 84–91]; an alternative demonstration occurs in I. J. Schwatt, *An Introduction to the Operations with Series*, Univ. of Pa. Press, 1924, p. 104.

In Sec. 48 of Jordan, *Calculus of Finite Differences*, occurs the probability problem, "If an urn contains m balls numbered 1, 2, $\cdots$, m, and if n balls are drawn with replacement, what is the probability that the sum of the numbers drawn shall be less than x?" In connection with the case $x=nm+1$ appears the identity

$$(-m)^n = \sum_{k=1}^{n}(-1)^k\binom{n}{k}\binom{km}{n},$$

which, for $m=n$, reduces to the result of the given problem.

Two Linear Simultaneous Systems

E 1248 [1957, 42]. *Proposed by Leo Moser, University of Alberta*

(a) The ten numbers $s_1 \leqq s_2 \leqq \cdots \leqq s_{10}$ are the sums of the five unknown numbers $x_1 \leqq x_2 \leqq \cdots \leqq x_5$ taken two at a time. Determine the x's in terms of the s's.

(b) Show that if $s_1 < s_2 < \cdots < s_6$ are six distinct numbers formed by taking the sums of four numbers two at a time, then there exist four other numbers which give the same sums when added in pairs.

Solution by C. F. Pinzka, Xavier University, Cincinnati, Ohio. (a) Obviously $s_1 = x_1 + x_2$, $s_2 = x_1 + x_3$, $s_9 = x_3 + x_5$, $s_{10} = x_4 + x_5$. Also $\Sigma s_i = 4\Sigma x_i$. These conditions suffice to give the solution

$$x_1 = s_1 + s_2 + s_{10} - (\Sigma s_i)/4,$$

$$x_2 = (\Sigma s_i)/4 - s_3 - s_{10},$$

$$x_3 = (\Sigma s_i)/4 - s_1 - s_{10},$$

$$x_4 = (\Sigma s_i)/4 - s_1 - s_9,$$

$$x_5 = s_1 + s_9 + s_{10} - (\Sigma s_i)/4.$$

(b) If one set is $\{x_i\}$, then another set is $\{S - x_i\}$, where $S = (\Sigma x_i)/2$. These will be distinct because $s_3 \neq s_4$ implies $S - x_4 \neq x_1$, where $S - x_4$ and x_1 are the least of their respective sets.

Also solved by J. L. Baker, Underwood Dudley, A. L. Epstein, N. J. Fine, Michael Goldberg, A. R. Hyde, V. F. Ivanoff, P. W. M. John, Joe Lipman, R. L. London, D. C. B. Marsh, L. A. Ringenberg, D. A. Robinson, Azriel Rosenfeld, E. D. Schell, and Gustavus Simmons. Late solution by A. S. Gregory.

Part (a) of the problem, with a solution, may be found in the Dec. 1955 issue of the *Newsletter* of the Canadian Mathematical Congress.

Cubes as Sums of Cubes

E 1249 [1957, 43]. *Proposed by C. M. Sandwick, Sr., Easton High School, Easton, Pa.*

Find an integer less than 1000, the cube of which may be represented as the sum of the cubes of three positive integers in five distinct ways.

Summary of the findings of Leon Bankoff, Los Angeles, Calif. Bankoff used the methods of Vieta, Euler, and Ramanujan to generate 216 primitive solutions of $a^3 + b^3 + c^3 = d^3$ $(d < 1000)$. From these he selected 74 suitable for forming imprimitives that could be collected into sets of five or more equal sums of 3 cubes. He ultimately found 47 integers that meet the requirements. The most striking example is 870^3, which can be expressed as the sum of 3 cubes in nine distinct ways:

$$870^3 = 17^3 + 687^3 + 694^3 = 537^3 + 564^3 + 687^3 = 235^3 + 485^3 + 810^3$$
$$= 200^3 + 540^3 + 790^3 = 260^3 + 550^3 + 780^3 = 380^3 + 480^3 + 790^3$$
$$= 225^3 + 630^3 + 735^3 = 330^3 + 450^3 + 810^3 = 435^3 + 580^3 + 725^3.$$

The other 46 integers are here grouped with regard to the number of ways their cubes can be expressed as the sum of 3 positive cubes:

8 ways: 348, 420, 522, 540, 630, 696, 720, 738, 756, 840, 900, 990.

7 ways: 174, 360, 450, 494, 792, 984.

6 ways: 246, 270, 396, 432, 504, 516, 580, 648, 810, 812, 864.

5 ways: 180, 216, 252, 261, 324, 369, 378, 600, 609, 618, 684, 690, 783, 820, 828, 972, 996.

Particular solutions were found also by M. Barnebey, W. J. Blundon, Michael Goldberg, John Leech, D. C. B. Marsh, Alan Wayne, and the proposer. Late solution by D. M. Brown.

Some allied references are: Heath, *Diophantus of Alexandria* (2nd ed.), p. 329 *et seq.*; Hardy and Wright, *An Introduction to the Theory of Numbers* (3rd ed.), p. 199 *et seq.*; Davenport, *The Higher Arithmetic*, p. 163; *Trans. Camb. Phil. Soc.* 22 (1920) 402; Dickson, *History of the Theory of Numbers*, vol. II, pp. 552–554; Dickson, *Introduction to the Theory of Numbers*, art. 36.

Centers of Similitude

E 1250 [1957, 43]. *Proposed by N. A. Court, University of Oklahoma*

Through a point G two secants GAD, GBC are drawn meeting a given circle (H) in the points A, D; B, C. Show that the points $E=(AB, CD)$, $F=(AC, BD)$ are the centers of similitude of the two circles orthogonal to (H) and having for centers the harmonic conjugates of G for the pairs of points A, D; B, C, respectively.

Solution by the proposer. The diagonal triangle EFG of the complete quadrangle $(Q)=ABCD$ inscribed in (H) is polar for (H) [Cremona, *Projective Geometry*, art. 260], and therefore EF is the polar of G for (H). Thus the points $I=(AD, EF)$, $J=(BC, EF)$ are separated harmonically from G by the pairs of points A, D; B, C, respectively, and the vertices E, F are conjugate with respect to circle (H).

Let (I), (J) be the circles drawn with I, J as centers orthogonal to (H). The circle (EF) having E, F for diametrically opposite points is also orthogonal to (H) [proposer's *College Geometry*, 2nd ed., art. 387]. We thus have three circles, (I), (J), (EF), orthogonal to the same circle (H) and having their centers on the same line $EIFH$, whence the three circles are coaxal [*ibid.*, art. 458]. Moreover, from quadrangle (Q) it results that the pairs of points I, J; E, F are harmonic, whence (EF) is the circle of similitude of circles (I), (J) [*ibid.*, art. 481], and therefore E, F are their centers of similitude.

Also solved by D. C. B. Marsh (analytically) and Beckham Martin (synthetically). Late solution by Victor Thébault.

Martin showed that if circles are drawn with centers at I and J and cutting (H) under angles α and β, respectively, then E, F will be the centers of similitude of these two circles if and only if $(\cos\alpha)/(\cos\beta)=IE/EJ$. It now follows that if $\alpha=\pi/2$, we must also have $\beta=\pi/2$.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscript should be typewritten, with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4748. *Proposed by Alexander Oppenheim, University of Malaya*

Suppose that x and y are defined by the two series

$$x = \frac{1}{a_1} + \frac{1}{a_1^2}\frac{1}{a_2} + \frac{1}{a_1^2 a_2^2}\frac{1}{a_3} + \cdots,$$

$$y = \frac{1}{a_1} + \frac{1}{a_1^3}\frac{1}{a_2} + \frac{1}{a_1^3 a_2^3}\frac{1}{a_3} + \cdots,$$

where the a_i are integers such that $a_i \geqq 2$ $(i=1, 2, \cdots)$. Prove that x and y are both rational or both irrational.

4749. *Proposed by Joseph Lehner, Los Alamos Scientific Laboratory*

Is it true that for every set E of positive measure contained in the interval $(0, 1)$, the numbers C_n have the property $|nC_n| < M$, $n=1, 2, \cdots$, where

$$C_n = \int_E e^{2\pi i n x} dx,$$

and M is a constant depending only on E?

4750. *Proposed by Leonard Carlitz, Duke University*

Find the denominator of

$$\binom{i}{m} = \frac{i(i-1)\cdots(i-m+1)}{m!}$$

where $i=\sqrt{-1}$ and the fraction is reduced to lowest terms in the field $R(i)$.

4751. *Proposed by Victor Thébault, Tennie, Sarthe, France*

If the Feuerbach hyperbola of a triangle ABC, with orthocenter H and incenter I, is tangent to the circle ABC, then it is also tangent to the circles BCH, CAH, ABH, and its focal axis is the perpendicular bisector of IH. (The Feuerbach hyperbola is the equilateral hyperbola determined by the four points, A, B, C, and I.)

4752. *Proposed by M. S. Klamkin, AVCO, Research and Development, Lawrence, Mass.*

Determine a set of n distinct, nonzero terms such that their geometric mean is the geometric mean of their arithmetic and harmonic means.

SOLUTIONS

Divergent Integral and Series

4670 [1956, 47, 190]. *Proposed by K. L. Chung, Syracuse University*

If $f(x)$ is continuous and nonnegative in $[0, \infty)$, and $\int_0^\infty f(x)dx = \infty$, then there exists an $h>0$ such that $\sum_{n=1}^{\infty} f(nh) = \infty$.

Editorial Note. As pointed out by the proposer, solution I [1957, 119] is incorrect because h there depends upon b and cannot be regarded as fixed while $b \to \infty$.

Minimal Weakly Prime Ideal

4691 [1956, 347]. *Proposed by R. E. Johnson, Smith College*

A weakly prime ideal of a ring R is any ideal I having the property that either $aR \subset I$ or $Ra \subset I$ implies that a is in I. Give an elementwise characterization of the unique minimal weakly prime ideal of R.

II. *Solution by the proposer.* In a former solution [1957, 375] it is claimed that a certain set $W(I)$ is weakly prime and that $aR \subset W(I)$ implies $a \in W(I)$. This is not so. From $aR \subset W(I)$ one can conclude that $R^n a r R^m \subset I$, where the integers m and n depend on r. It can happen that no m and n work for all $r \in R$. A valid solution may be obtained as follows.

If $\{x\} = \{x_1, x_2, \cdots, x_n, \cdots\}$ is a sequence of elements in R, let

$$x^n = x_1 \cdot x_2 \cdot \cdots \cdot x_n, \qquad {}^n x = x_n \cdot \cdots \cdot x_2 \cdot x_1.$$

An element a in R is called *strongly nilpotent* if for every pair $(\{x\}, \{y\})$ of sequences in R there exist integers m and n such that ${}^m x a y^n = 0$. We prove the

THEOREM. *The unique minimal weakly prime ideal of R is the set S of all strongly nilpotent elements of R.*

We first prove the following lemma.

LEMMA. An element a of R is in S if and only if $aR \subset S (Ra \subset S)$.

Proof. If a is in S and r is in R, and if $(\{x\}, \{y\})$ is a pair of sequences in R, then let $\{z\} = \{r, y_1, \cdots, y_n, \cdots\}$. By definition of S, there exist integers m and n such that ${}^m x a z^n = 0$. Hence, also, ${}^m x (ar) y^n = 0$. We conclude that ar is in S for every r in R.

Conversely, if ar is in S for every r in R, and if $(\{x\}, \{y\})$ is a pair of sequences in R, then let $\{z\} = \{y_2, y_3, \cdots, y_n, \cdots\}$. By definition of S, there exist integers m and n such that ${}^m x (a y_1) z^n = 0$. Hence ${}^m x a y^{n+1} = 0$ and we conclude that a is in S.

Proof of theorem. It is clear that S is closed under addition and, in view of the lemma, that S is a weakly prime ideal of R.

If I is any weakly prime ideal of R and a is not in I, then $RaR \not\subset I$. Hence there exist elements x_1, y_1 in R such that x_1ay_1 is not in I. By the same reasoning, there exist elements x_2, y_2 in R such that $x_2x_1ay_1y_2$ is not in I. Continuing, there exists a pair $(\{x\}, \{y\})$ of sequences in R such that ${}^m xay^n \neq 0$ for all integers m and n. Clearly a is not in S, and we conclude that $S \subset I$.

Inferior and Superior Limits

4692 [1956, 347] Corrected. *Proposed by A. E. Currier, United States Naval Academy*

Given $f(x)$ defined by

$$f(x) = -\frac{1}{2} + \sum_{n=0}^{\infty} (-1)^n x^{2n},$$

prove that

$$(1) \qquad -\frac{3+8\sqrt{6}}{750} < \liminf_{x \to 1^-} f(x) < -\frac{1}{750},$$

$$(2) \qquad \frac{1}{750} < \limsup_{x \to 1^-} f(x) < \frac{3+8\sqrt{6}}{750}.$$

Editorial Note. The proposer effects the solution of the problem by an approximation of Cesàro means of the third order of $\sum a_r$ defined by $f(x) = \sum_{r=0}^{\infty} a_r x^r = -\frac{1}{2} + \sum_{n=0}^{\infty} (-1)^n x^{2^n}$. The somewhat lengthy details are available for any interested reader.

An Extremal Property of the Normal Distribution

4699 [1956, 497]. *Proposed by I. J. Schoenberg, University of Pennsylvania*

Let $f(x)$ be a frequency function of finite variance σ^2 and such that $f(x) \cdot \log f(x)$ is summable ($x \log x$ is defined to be $=0$ if $x=0$). Show that

$$\int_{-\infty}^{\infty} f(x) \log f(x) dx \geqq \log \frac{1}{\sigma\sqrt{2\pi e}}$$

with equality if and only if $f(x)$ is equal almost everywhere to a normal function

$$\frac{1}{\sigma\sqrt{2\pi}} e^{-(x-m)^2/2\sigma^2}.$$

Solution by Edgar Reich and Chih-yi Wang, University of Minnesota. Since $f(x)$ is a frequency function of finite variance σ^2 we have

$$f(x) \geqq 0, \quad \int_{-\infty}^{\infty} f(x)dx = 1, \quad \int_{-\infty}^{\infty} xf(x)dx = m, \quad \int_{-\infty}^{\infty} (x-m)^2 f(x)dx = \sigma^2.$$

Then by the familiar inequalities for the arithmetic, geometric and harmonic means, we have for any function $g(x)$, $g(x) \geqq 0$.

$$(1) \qquad \left[\int_{-\infty}^{\infty} \frac{f(x)}{g(x)}\, dx\right]^{-1} \leqq \exp\left\{\int_{-\infty}^{\infty} f(x) \log g(x) dx\right\} \leqq \int_{-\infty}^{\infty} f(x)g(x)dx.$$

Let the normal function of the given form be denoted by $N(x)$. Then by substituting either $g(x) = f(x)/N(x)$ in the left inequality of (1) or $g(x) = N(x)/f(x)$ in the right inequality of (1), we obtain a result from which the required inequality follows easily. Equality holds if and only if $g(x) = C$, a constant, almost everywhere. But by the relations of normalization

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} N(x)dx = 1,$$

we find $C = 1$.

Also solved by N. J. Fine, C. H. Kraft and Ingram Olkin, Morton Kupperman, the proposer, and one whose solution is unsigned.

Editorial Note. The problem is given, with indicated solution, in Woodward, *Probability and Information Theory, with Applications to Radar*, p. 25. The negative of the integral in the above proposal is termed the entropy of the distribution given by $f(x)$. For a closely related problem see Moriguti, *A lower bound for the probability moment of any absolutely continuous distribution with finite variance*, Ann. Math. Statist., vol. 23, 1952, pp. 286–289.

Power Series with Preassigned Zero

4701 [1956, 498]. *Proposed by Paul Erdös, Technion Mathematics Department, Haifa, Israel*

Let α be complex, $|\alpha| < 1$. Prove that there exists a power series $\sum a_k z^k$ with rational integral coefficients, $|a_k| \leqq [1/|\alpha|^2]$, such that $\sum a_k \alpha^k = 0$.

Solution by the proposer. Write $\beta = [1/|\alpha|^2]$, and assume that $a_k \neq 0$ for some values of k. Consider the $(\beta+1)^n$ polynomials $\sum_{k=0}^{n-1} a_k z^k$, $0 \leqq a_k \leqq \beta$. Clearly $|\sum a_k z^k| < C$ for $|z| \leqq 1$. We can assume that all the $(\beta+1)^n$ points $\sum_{k=0}^{n-1} a_k \alpha^k$ are distinct, for otherwise there would be a polynomial $\sum b_k \alpha^k = 0$ with $|b_k| \leqq \beta$. Now we use the lemma: *If there are n distinct points* $|z_i| \leqq C_1$, $1 \leqq i \leqq n$, *then there are two of them* z_{i_1} *and* z_{i_2} *for which* $|z_{i_1} - z_{i_2}| < C_2 n^{-1/2}$. (To see this, consider circles of center z_i and radius $\frac{1}{2}C_2 n^{-1/2}$; for large C_2 they have to overlap.) From the lemma it follows that there exist sets of coefficients a_k, $a_k' \leqq \beta$ such that

$$\sum_{k=0}^{n-1} (a_k - a_k')\alpha^k < C_3/(\beta + 1)^{n/2}.$$

Let l be the smallest subscript with $a_l \neq a_l'$, then upon dividing through by α^l we obtain a polynomial $\sum_{r=0}^{n-1} b_r z^r$, $b_0 = a_l - a_l' \neq 0$, for which

$$\sum_{r=0}^{n-1} b_r \alpha^r < \frac{C_3}{\alpha^l(\beta + 1)^{n/2}} < \frac{C_3}{\alpha^n(\beta + 1)^{n/2}}.$$

Now since $\beta+1>1/|\alpha|^2$ we have $C_3/\alpha^n(\beta+1)^{n/2}\to 0$. Thus it follows that for every ϵ there exists an n_0 so that for every $n>n_0$ there exists a polynomial $|\sum_{r=0}^{n-1} b_r\alpha^r|<\epsilon$, $|b_r|\leqq\beta$.

Now let $f_n(z)$ be a sequence of such polynomials with $|f_n(\alpha)|\to 0$ as $n\to\infty$. A power series $\sum c_k z^k$ will be called good if every partial sum $\sum_{k=1}^{m} c_k z^k$ is the beginning of infinitely many polynomials $f_n(z)$, *i.e.*, for infinitely many n, $f_n(z)=c_0+c_1z+\cdots+c_mz^m+\cdots$. Since there are only finitely many polynomials $\sum_{k=0}^{m} c_k z^k(|c_k|\leqq\beta)$ and infinitely many polynomials $f_n(z)$, a good power series exists from the convergence of $\sum_{k=0}^{\infty} c_k\alpha^k$; it then follows that $\sum_{k=0}^{\infty} c_k\alpha^k=0$.

Editorial Note. H. Graetzer proved this result (Journal of the London Mathematical Society, (1947), pp. 90–92) without demanding that the a_k be bounded. E. G. Strauss has a simple proof for the case $1/\alpha^2$ integral. L. Carlitz has submitted a short proof under the hypothesis that $|a_k|\leqq 2|\alpha|^{-k}$. For a related result see problem no. 4498 [1953, 634].

A Doubly Infinite Array of Sets

4702 [1956, 498]. *Proposed by D. J. Newman, Advanced Development Division, AVCO, Lexington, Mass., and W. E. Weissblum, Massachusetts Institute of Technology*

Let

$$A_{11}, A_{12}, A_{13}, \cdots$$

$$A_{21}, A_{22}, A_{23}, \cdots$$

$$\cdots\cdots\cdots$$

be an infinite square array of sets of real numbers in (0, 1). If $A_{kn}\to 0$ for each fixed k, does it follow that there exists a diagonal sequence, $A_{1n_1}, A_{2n_2}, A_{3n_3}, \cdots$, whose limit is 0?

Solution by Jerome Keisler, California Institute of Technology. The answer is no, and I shall proceed to construct an array in which there exists no diagonal sequence whose limit is the null set 0. Without loss of generality we may take the $A_{i,j}$ to be sets of real numbers in the closed interval $I_0=\{a\leqq x\leqq b\}$, where $0<a<b<1$, with a, b constant.

For any closed interval $I=\{c\leqq x\leqq d\}$, define $H_1(I)$ as the closed interval $\{c\leqq x\leqq(c+d)/2\}$, *i.e.*, the left half of I. Define $H_2(I)=H_1(I-H_1(I))$, and in general $H_n(I)=H_1(I-\bigcup_{i=1}^{n-1} H_i(I))$. We have $H_n(I)\subset I$. The sequence $H_1(I)$, $H_2(I)$, $\cdots$, $H_n(I)$, $\cdots$ consists of non-overlapping closed intervals, and any real point p lies in at most two of these intervals. Finally if I is the union of the closed intervals $I_j=\{a_j\leqq x\leqq b_j\}$, such that $a_1<b_1<a_2<b_2<\cdots<a_j<b_j<\cdots$, define $H_n(I)=\bigcup_j H_n(I_j)$.

Now take as the first-row sequence of sets $A_{1,j}$, the intervals $A_{1,n}=H_n(I_0)$. In general, take as the kth row sequence $A_{k,j}$, the sets $A_{k,n}=\bigcup_j H_n(a_{k-1,j})$. The infinite square array of sets $A_{i,j}$ is now inductively defined. For each k, we have

$\lim_n A_{k,n}=0$ since each real point lies in at most two $A_{k,n}$.

Now consider any subsequence A_{1,n_1}, A_{2,n_2}, $\cdots$. Define $S_1=A_{1,n_1}$, $S_2=H_{n_2}(A_{1,n_1})\subset A_{2,n_2}$, and, in general, $S_k=H_{n_k}(S_{k-1})\subset A_{k,n_k}$. We thus have a sequence $\{S_i\}$ with the properties for all k,

1) $S_k \subseteq A_{k,n_k}$, 2) $S_k \subset S_{k-1}$ $(k \geqq 2)$,

3) S_k is a closed interval, 4) the length of $S_k \to 0$ as $k \to \infty$.

Thus S_k is a sequence of nested intervals, and there exists a unique point q which is in every S_k. Then q is in every A_{k,n_k}, and $q \in \lim_k A_{k,n_k}$. Then $\lim_k A_{k,n_k} \neq 0$ for any subsequence A_{k,n_k}.

Also solved by L. A. Rubel, B. I. Penkov and the proposer.

RECENT PUBLICATIONS

EDITED BY RICHARD V. ANDREE, University of Oklahoma

All books for review should be sent directly to R. V. Andree, Department of Mathematics, University of Oklahoma, Norman, Oklahoma, and not to any of the other editors or officers of the Association.

Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, vol. II: Contributions to Probability Theory. Ed. by Jerzy Neyman. University of California Press, Berkeley and Los Angeles, 1956. x+246 pp. $6.50.

This book is the second of five volumes of papers contributed to the Third Berkeley Symposium, held under the auspices of the Statistical Laboratory and the Department of Statistics of the University of California at Berkeley. Most of the papers in this volume were presented in sessions held during July and August, 1955. The contributions were at the invitation of the Statistical Laboratory.

The present volume consists of 15 articles by a group of eminent scholars. No effort was made to unify the topics treated, but rather the papers represent current individual investigations undertaken by the authors. The result is a very good cross-section of modern probability theory as it is unfolding today, and in particular reflects strikingly the diminishing interest in limit theorems associated with sums of random variables and the vigorous growth of stochastic processes in general.

It is of course impossible to give here a review of each of the papers individually—they will undoubtedly be reviewed separately in, *e.g.*, *Mathematical*

Reviews. The following is a reproduction of the table of contents:

David Blackwell, *On a Class of Probability Spaces*, pp. 1–6.
S. Bochner, *Stationarity, Boundedness, Almost Periodicity of Random Valued Functions*, pp. 7–28.
K. L. Chung, *Foundations of the Theory of Continuous Parameter Markov Chains*, pp. 29–40.
A. H. Copeland, Sr., *Probabilities, Observations and Predictions*, pp. 41–48.
J. L. Doob, *Probability Methods Applied to the First Boundary Value Problem*, pp. 49–80.
R. M. Fortet, *Random Distributions with an Application to Telephone Engineering*, pp. 81–88.
J. M. Hammersley, *The Zeros of a Random Polynomial*, pp. 89–112.
T. E. Harris, *The Existence of Stationary Measures for Certain Markov Processes*, pp. 113–124.
Kiyosi Ito, *Isotropic Random Current*, pp. 125–132.
Paul Lévy, *A Special Problem of Brownian Motion, and a General Theory of Gaussian Random Functions*, pp. 133–176.
Michel Loeve, *Ranking Limit Problem*, pp. 177–194.
Eugene Lukacs, *Characterization of Populations by Properties of Suitable Statistics*, pp. 195–214.
Karl Menger, *Random Variables from the Point of View of a General Theory of Variables*, pp. 215–230.
Edith Mourier, *L-random Elements and L*-Random Elements in Banach Spaces*, pp. 231–242.
R. Salem and A. Zygmund, *A Note on Random Trigonometric Polynomials*, pp. 243–246.

DONALD A. DARLING
University of Michigan

Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, vol. IV: Contributions to Biology and Problems of Health. Ed. by Jerzy Neyman. University of California Press, Berkeley and Los Angeles, 1956. vii+179 pp. $5.75.

This book is one of five volumes reporting the proceedings of a symposium held during December 1954 and July–August 1955. The material contained in vol. IV may best be summarized by reproducing the table of contents:

I. Contributions to Biology
James Crow and Motoo Kimura, *Some genetic problems in natural populations.*
Everett R. Dempster, *Some genetic problems in controlled populations.*
Jerzy Neyman, Thomas Park and Elizabeth L. Scott, *Struggle for existence. The Tribolium model: Biological and statistical aspects.*

II. Contributions to Problems of Health

M. S. Bartlett, *Deterministic and stochastic models for recurrent epidemics.*

A. T. Bharucha-Reid, *On the stochastic theory of epidemics.*

Chin L. Chiang, J. L. Hodges, Jr., and J. Yerushalmy, *Statistical studies in medical diagnoses.*

Jerome Cornfield, *A statistical problem arising from retrospective studies.*

David G. Kendall, *Deterministic and stochastic epidemics in closed populations.*

William F. Taylor, *Problems in contagion.*

Of special interest to mathematicians is the range of mathematical concepts and techniques employed by the several authors. The following were noted (and others were no doubt used if not simply missed by the reviewer): partial differentiation, differential equations, Laplace transforms, eigenfunctions, difference equations, probability and distribution theory, Markoff chains, Monte Carlo methods, method of least squares, statistical theory of estimation and hypothesis testing, decision functions, stochastic processes.

No brief review can do justice to the skill and imagination shown by the authors in applying the techniques mentioned above to the difficult problems encountered in the biological sciences and in the allied fields of health and medicine. It is not that the techniques themselves are difficult, but rather that the expression of the problems in mathematical terms is not easy. Perhaps the greatest contribution of the mathematicians and statisticians has been in helping the biologist state his problems so that a mathematical formulation is possible. This reformulation of the problems in mathematical language results in two major benefits: (1) Once the problems are expressed in mathematical terms, solutions are frequently easily obtained. Of course, in so expressing the problems, simplifications, desirable or otherwise, have often been made. (2) The research workers in biology, medicine and public health have been forced to re-examine their assumptions and theories in the light of the questions posed by the mathematicians and statisticians.

It is gratifying to note that mathematics and mathematicians are aiding in the biological sciences. This reviewer has long thought that mathematicians have concerned themselves too much with the physical sciences and too little with the biological sciences. In recent years, however, as shown by the volume being reviewed here and other publications, it is evident that mathematicians have become aware of the varied opportunities to apply their knowledge in an area previously ignored. It is hoped that the degree of cooperation between the researcher in the biological sciences and the mathematician will continue to grow in the years ahead.

BERNARD OSTLE
Montana State College

Elements of Pure and Applied Mathematics. By Harry Lass. New York, International Series in Pure and Applied Mathematics, McGraw-Hill, 1957; ix +491 pp. $7.50.

This book is an outgrowth of courses which the author has taught, and is designed so that it may be used as a textbook in various courses from the junior level onward.

The material covered is indicated by the titles of the ten chapters. These are: (1) Linear equations, determinants, and matrices, (2) Vector analysis, (3) Tensor analysis, (4) Complex variable theory, (5) Differential equations, (6) Orthogonal polynomials, Fourier series, and Fourier integrals, (7) The Stieltjes integral, Laplace transforms, and calculus of variations, (8) Group theory and algebraic equations, (9) Probability theory and statistics, (10) Real variable theory.

Flexibility for the use of the book as a text is provided by the independence of the chapters; this is not strictly true of the last chapter since frequent reference to it is made in the rest of the work. Exercises are provided at the end of sections. The comprehensive choice of topics is a good one. As is, however, apt to be the case in a work of such extensive scope, some topics are treated more briefly than one might like. An early example of this is the omission of the Laplace development by minors in the treatment of determinants; the reader instead is urged to read this topic in a conventional book. Even so, the definition of determinants using permutations finds so little application either for hand or machine calculation of determinants, or for deducing theorems, that the recursive definition of a determinant in terms of its minors or cofactors would seem preferable.

Throughout the book, references are made to the last chapter, that dealing with real variable theory. This chapter consists of a development of the real number system beginning with the Peano postulates, an introduction to the theory of sets of real numbers, to cardinal numbers, a discussion of limits, continuity, and topics generally treated in courses in advanced calculus. It is regrettable that rigor appears lacking at some crucial points. We give two examples: The field of rational numbers is derived from the ring of integers by postulating a *rational function*, instead of the construction using ordered pairs. It would appear that this is not a suitable substitute, as it more or less begs the question. In order to postulate a function, we must assume its range, in this case the rational numbers themselves. Limits are treated in section 10.12 by loosely introducing an infinitesimal as "a variable which approaches zero as a *limit*." The discussion of limits is then carried on in terms of infinitesimals. From this discussion the reader would have difficulty in giving valid proofs of the fundamental limit theorems, as he is directed to do. An acceptable definition of continuity follows the treatment of limits.

A. A. Grau
University of Oklahoma and
Oak Ridge National Laboratory

History of Analytic Geometry. By C. B. Boyer, Scripta Mathematica, New York, 1956, ix+291 pp. $6.00.

Here is an integrated survey of the history of analytic geometry from the earliest glimmerings in ancient Mesopotamia and Egypt up through the death in 1868 of the subject's most powerful and prolific contributor, Julius Plücker. Leaning largely upon the earlier, but either incomplete or inconveniently inaccessible, works of Loria, Tropfke, Wieleitner, and Coolidge, Professor Boyer has in scholarly fashion sketched the fascinating story of the emergence and development of this important branch of mathematics, filling in many lacunae and correcting many errors perpetrated by early writers and perpetuated by subsequent historians of mathematics.

There is no unanimity of opinion among historians of mathematics as to who invented analytic geometry, nor even as to what age should be credited with the invention, and this matter certainly cannot be settled without an agreement as to the defining characteristics of the subject. Professor Boyer steers an admirably restrained course here, and though he bespeaks his own views he gives a fair account of existing opinions. A bird's-eye view of the book's chronological trend is obtained from the titles of the nine chapters: I, The Earliest Contributions; II, The Alexandrian Age; III, The Medieval Period; IV, The Early Modern Prelude; V, Fermat and Descartes; VI, The Age of Commentaries; VII, From Newton to Euler; VIII, The Definitive Formulation; IX, The Golden Age. These chapters are sown with 470 footnotes of considerable worth to the more deeply interested reader. At the end of the book appears a list of 76 important primary sources (arranged roughly in chronological order) and a list of 142 secondary works (arranged alphabetically by authors). The value of these references has been enhanced by accompanying each with a brief indication of the nature of the material involved. The book closes with a good index.

Professor Boyer's book is published as Numbers Six and Seven of the Scripta Mathematica Studies, and thus enjoys the neat appearance for which these studies are known. Some readers of the book will recall that goodly portions of the manuscript appeared earlier in the pages of *Scripta Mathematica*. It hardly seems fair to attempt a criticism of the book in regard to areas of omission, for it must be admitted that Professor Boyer has well chosen his material so as not to exceed a volume of modest size. Nor would it be fitting to comment on observed typographical errors; they are not serious and are well within expected limits.

Teachers and students of analytic geometry, and all interested in the history of mathematics, owe Professor Boyer a very real debt of gratitude for performing the patient, painstaking, and time consuming research that undoubtedly went into the preparation of this fine book.

HOWARD EVES
University of Maine

Advanced Real Calculus. By Kenneth S. Miller. Harper, New York, 1957. viii+185 pp. $5.00.

The book is directed principally at the junior or senior mathematics major. The student is led from the Dedekind cut through the notions of differentiation and Riemann integration of functions of one variable to the corresponding notions for functions of two variables.

Concentration is on the ϵ, δ method with complete details of the method presented in almost all instances. Unfortunately, while stressing the ϵ, δ method, there is a lack of precision with several other notions. For example, a function of one real variable is described but not defined, while no attempt is made to define a function of two real variables. A "number" is a Dedekind cut of the rational numbers. However, with no further discussion of isomorphism or identification, certain "numbers" are called rational numbers. The conception of mathematical elegance that is given the student is that of showing that a quantity is less than ϵ, rather than a fixed multiple of ϵ.

A novel feature of the book is that complete solutions to many of the exercises in the book are given in an appendix.

PAUL CIVIN
University of Oregon

The Leibniz-Clarke Correspondence, together with Extracts from Newton's *Principia* and *Opticks*, edited with Introduction and Notes by H. G. Alexander. Philosophical Library, New York, 1956. 200 pp. $4.75.

This correspondence exchanged during the years 1715–1716, at the suggestion of Caroline, Princess of Wales, in which Samuel Clarke acted as the authorized champion of Sir Isaac Newton, is one of the most important controversies in the history of modern philosophy and science. First published by Clarke in 1717 in a bilingual edition for which he had translated Leibniz's French papers into English, the complete English text is here reprinted for the first time in a separate and extremely careful and competent edition, accompanied by extracts from Newton's and Leibniz's writings which have a direct bearing on the problems under discussion.

So fundamental are the issues involved in this conflict between two thinkers of equal stature that they have continued to be debated up to the present time. The nature of space and time and the measurement of force, together with many correlated questions, are thoroughly discussed in these letters by philosopher-scientists, of whom each fought from inside his own majestic system of physics, mathematics, and metaphysics. No understanding, compromise, or mutual approach could therefore appear possible, when Leibniz's death put an end to the correspondence. But the later development of science, at least as long as classical

mechanics dominated the scene, seemed to decide in favor of the Newtonian system which eclipsed Leibniz's idealistic solution during almost two centuries.

In his introduction, H. G. Alexander has masterfully epitomized the conflicting doctrines and traced the development of the problems involved through the contributions of Berkeley, Maclaurin, Euler, Boscovitch, Kant, and other scientists up to Ernest Mach and Einstein. His conclusion stating that "in the light of modern physics, it is perhaps best to call it [the controversy] a drawn contest," can hardly be challenged by the unpredjudiced student who will appreciate this authoritative guidance in the study of an historically and systematically central issue.

PAUL SCHRECKER
University of Pennsylvania

Numerical Integration of Differential Equations. By A. E. Bennett, W. E. Milne, and H. Bateman, Dover, New York, 1956, 108 pp. $1.35.

This little book is a reproduction of the monograph, of the same title, prepared in 1931 for the National Research Council, with no addition or deletion of material.

Much has happened in this field since 1931, including and in particular the development of high-speed calculating machines. There is little here of interest to the "programmer," and no guidance at all respecting practical methods for solving the various classes of partial differential equations. Chapter 4, devoted to this topic, is in fact mainly restricted to a brief account of classical semi-analytic methods such as those of Rayleigh and Ritz and "least squares," the use of Fourier series, and some comments on integral equations. Chapter 2, similarly, has little of immediate practical value, being an historical account of methods of successive approximation and a justification of step-by-step methods of integration. Neither of these chapters gives worked examples.

Chapters 1 and 3, however, have much useful information in little space. In Chapter 1 are collected nearly all the useful finite-difference formulae for interpolation, differentiation and integration, both with differences and with ordinates only, and with valuable comments on their use and misuse. Chapter 3 gives almost all known methods, with numerical illustration, for step-by-step solution of ordinary differential equations.

For the scholar, and particularly the historian, there is a wealth of information, including some five hundred references of original material and additional reading! It is doubtful whether many readers will take advantage of this opportunity to delve into the past, but Chapters 1 and 3 probably justify the rather modest investment in this slim volume.

L. Fox
University of California, Berkeley

Numbers: Fun and Facts. By J. Newton Friend. Charles Scribner's Sons, New York, 1954. xi+208 pp. $2.95.

This is a book whose title adequately describes the contents. The only requirements for reading and enjoying it are a knowledge of elementary arithmetic and a fascination for problem solving. The purpose of the book as given in the preface is, "to show how interesting, indeed fascinating, is the study of numbers, their origin, and peculiarities, and of the traditions, legends and superstitions that have in the course of ages, collected around them."

The author discusses: one-to-one correspondence, counting by various methods, Roman and Hindu-Arabic numerals, origin of numerical symbols, kinds of numbers and superstitions about various numbers. There is also a short description of gematria and numerology. The discussions of one-to-one correspondence, concrete and abstract numbers, the number π and number sense are very well done.

There are many problems in this attractive book. In the prologue the author states that several of them are from what appears to be a 17th century monastic manuscript that he found in an old shop. There are problems based on different ways of writing numbers including mirror writing and palindromes. There are match stick, coin and age problems. One chapter consists of problems where the digits are replaced by other symbols. The author gives an analysis of the solution for several problems.

There are a few typographical errors, one wrong answer and several incomplete answers. One will be surprised to read on p. 25, "The sole purpose of the 0 (in 50) is to show that 5 occupies the second place." On p. 69, 0/0 is treated as an absurdity instead of an indeterminate form.

MARION P. EMERSON
Southwest Missouri State College

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items should be submitted at least two months before publication can take place.

THIRD U. S. NATIONAL CONGRESS OF APPLIED MECHANICS

The third U. S. National Congress of Applied Mechanics will be held at Brown University, Providence, Rhode Island, June 11–14, 1958. All research workers in the field are cordially invited to submit papers constituting original experimental or theoretical contributions to Applied Mechanics, including mechanics of rigid bodies and deformable solids, mechanics of fluids and gases, thermodynamics and heat transfer. Instructions to authors of papers may be obtained from the Organizing Committee. It is expected

that papers accepted by the Editorial Committee with the advice of recognized authorities and presented at the Congress, will be published in full in the Proceedings of the Congress.

To be considered for presentation at the Congress, complete papers and manuscripts must be submitted to the Chairman of the Editorial Committee before January 1, 1958; to be scheduled for presentation the final manuscript of a paper must have been accepted before May 1, 1958. To avoid delays caused by overburdening reviewers and editorial staff, authors are urged to submit manuscripts well ahead of the deadline of January 1, 1958.

The papers will be grouped by subject, and 30 minutes will be allotted for presentation and discussion of each paper. Arrangements will also be made for general lectures by outstanding authorities on subjects of general interest to members of the Congress. Facilities will be provided for informal discussions and social contact.

Inquiries regarding the Congress should be addressed to D. C. Drucker, Secretary of the Congress, Brown University, Box D, Providence 12, R. I.

POLISH SCHOLARSHIPS FOR FOREIGN STUDENTS

Poland's Ministry of Higher Education is offering a number of scholarships at Polish universities for qualified foreign graduate students. The awards, which run from six months to two years, will be made for advanced study and research in mathematics, Polish linguistics and literature and Polish history. Slavic philology is included in the program. Highly qualified advanced undergraduates may also apply for the awards.

The Polish scholarships include double-room dormitory accommodations and a monthly sum of about 2400 zlotys. This sum is adequate to cover living costs, tuition and ordinary personal expenses. The awards do not cover transportation to and from Poland although in exceptional cases assistance may be given towards the cost of the return trip. If such aid is required, it must be indicated in the scholarship application.

Application forms are available from the Cultural Attaché of the Polish Embassy, 2640 Sixteenth St., N.W., Washington 9, D. C. Applications should be accompanied by a transcript of the candidate's university or college record, a biographical sketch, a statement of chief academic interests and of the work the applicant hopes to do in Poland. In addition, there should be a letter of recommendation from a faculty member under whom the applicant has done his major course work, copies of diplomas or academic certificates and three recent passport-size photographs.

OPPORTUNITY FELLOWSHIPS OF JOHN HAY WHITNEY FOUNDATION

The competition for Opportunity Fellowships is open to any citizen of the United States (including residents of territories) who has given evidence of special ability and who has not had full opportunity to develop his talents because of arbitrary barriers, such as racial or cultural background or region of residence. Candidates are expected to be mature enough to have given *positive evidence of superior promise*, yet young enough to have their careers before them; in general to be between the ages of 22 and 35 and to have completed their general education. While the Committee of Award has full discretion to take all factors into account and make awards outside the above ages and qualifications, candidates under 35 are given decided preference.

The fellowships are open not only for academic study (graduate) but for any kind of training or experience (journalism, industry, labor, the arts, *etc.*) which may be most useful in developing varied talents and varied forms of leadership.

Awards are expected normally to range from $1,000 to $3,000 depending on the nature of the proposed project and the financial need of the candidate. It is hoped that in many cases funds from other sources may supplement these awards (for example fellowships for study in foreign countries, additional scholarship aid from universities, pay-

ments for certain types of apprentice work, *etc.*). Awards are for a full year of serious work, *not for incidental or temporary projects*. In special cases grants may be renewed for a second year or more.

Awards are made annually by a special Committee on the basis of formal written applications by the candidates on forms provided by the John Hay Whitney Foundation. Completed applications must be filed not later than November 30 so as to assure ample time for processing applications, assembling references, and making selections. Awards are announced in April or May. Communications should be addressed: Opportunity Fellowships, John Hay Whitney Foundation, 630 Fifth Ave., New York 20, New York.

RESEARCH POTENTIAL AND TRAINING IN THE MATHEMATICAL SCIENCES

The Committee on the Survey announces the completion of the SURVEY OF RESEARCH POTENTIAL AND TRAINING IN THE MATHEMATICAL SCIENCES. The Survey began in January 1955, and nearly thirty mathematicians have served on its Committee and four Subcommittees during the past two and one-half years.

The Survey has produced three documents. One of these is the Report on a Conference on Undergraduate Mathematics Curricula. The Conference was held at Hunter College on October 12–13, 1956 and was attended by 25 representatives of various colleges and universities. There were formal reports by various speakers covering many aspects of, and subjects related to, undergraduate education in mathematics, and these formal reports have been incorporated into the conference report. This document has been circulated widely. A small number of copies have been deposited with the National Science Foundation and are available on request.

The other two documents constitute the Final Report. Part I is a 163 page report on the organization of the Survey and the results of two large scale data-gathering activities of the Committee. The first activity consisted of interviews of the principal Ph.D. granting institutions. Fifty-nine departments of mathematics were visited, and three others completed the interview schedule without a visit. The data presented as obtained by these interviews are concerned with such matters as facilities, libraries, promotion and leave policies, administration, and salaries. The remainder of the report presents a set of tables giving the composite life story of 1851 mathematicians who received the Ph.D. from 1915 to 1954, as derived from their replies to a very penetrating questionnaire.

Part II of the Final Report consists of the reports of the four Subcommittees, and the 28 resulting recommendations of the Committee. We note that the Subcommittee on Undergraduate Colleges recommended the extended support of prize competitions in mathematics, urged that an attempt be made to call to the attention of industry the need for support of mathematics in undergraduate colleges, and requested that the National Science Foundation publish annually an up-to-date record of the mathematics faculties of all colleges and universities with the degrees held, and the date and place these degrees were conferred.

The report of the Subcommittee on Research Environment discussed such matters as facilities, salaries, leaves and fellowships, and made a number of recommendations on these matters.

The Subcommittee on Non-Teaching Opportunities recommended the exploration of further methods for establishing understanding among academic mathematicians of the nature of non-academic mathematics and made recommendations on various other related matters. The report also called attention to the recent action of the American Mathematical Society in adding the SIAM Journal to the group of journals receiving a subsidy from the American Mathematical Society, and suggested the possibility that the Journal may provide another desirable publication outlet for some of the research results of non-academic mathematicians.

The final report, from the Subcommittee on Publications, recommended the establishment of a journal for expository articles, and the establishment of prizes for mathematical books, recommended the subsidization of mathematical publications by the National Science Foundation on a permanent basis, and recommended that the volume of translations published by the American Mathematical Society be at least doubled.

The Final Report has been distributed to the chairmen of about 110 departments of mathematics, to Committee and Subcommittee members, and to the officers of various mathematical organizations. A limited number of additional copies are available, and can be obtained by writing to the Program Director for Mathematics, National Science Foundation, 1520 H Street, Washington 25, D. C.

A. A. ALBERT, *Chairman, Committee on the Survey*

PERSONAL ITEMS

Professor H. W. Brinkmann, Swarthmore College, was the representative of the Association at the inauguration of President Frederick de Wolfe Bolman, Jr., of Franklin and Marshall College on April 6, 1957.

Professor G. K. Kalisch, University of Minnesota, represented the Association at the inauguration of the Very Reverend James P. Shannon as President of the College of St. Thomas on May 8, 1957.

Professor Marguerite Lehr, Bryn Mawr College, was the representative of the Association at the Convocation commemorating the Hundredth Anniversary of the National Education Association in Philadelphia, Pennsylvania, on July 3, 1957.

Professor S. S. Wilks, Princeton University, represented the Association at the sesquicentennial dinner of John Wiley and Sons, Inc. in New York City on May 28, 1957.

Dr. Max Herzberger, Head, Optical Research, Eastman Kodak Company, Rochester, New York, has been honored by election to the Bavarian Academy of Science.

Professor Emeritus W. C. Krathwohl, Director of Tests, Institute for Psychological Services, Illinois Institute of Technology, received an Alumni Distinguished Service Award on May 3, 1957.

Catholic University of America: Mr. D. B. Tomiuk, Graduate Assistant, has been appointed Instructor; Associate Professor J. N. Rice has retired; Dr. R. G. Laha, Indian Statistical Institute, Calcutta, has been appointed Research Associate; Professor Eugene Lukacs who is on leave of absence is lecturing at the Sorbonne, Paris, France.

Fresno State College: Mrs. Margaret Finn, University of Southern California, Assistant Professors W. C. Guenther, Arizona State College, Tempe, J. G. Marica, Humboldt State College, E. S. Robbins, Arizona State College, Tempe, and Mr. Peter Yff, University of Illinois, have been appointed Assistant Professors; Dr. S. J. Bryant, Mr. D. J. Ewy, Dr. V. E. Howes, and Dr. T. C. Kipps have been promoted to Assistant Professors.

Los Angeles City College: The Seventh Annual William B. Orange Mathematics Prize Competition for students in Los Angeles high schools was held on May 17, 1957 with 156 students from 33 local high schools participating.

Massachusetts Institute of Technology: Dr. H. P. McKean, Jr., Princeton University Dr. M. L. Minsky, Society of Fellows, Harvard University, and Assistant Professor D. B. Ray, Cornell University, have been appointed Assistant Professors; Mr. R. T. Dames, Willow Run Research Center, University of Michigan, Mr. A. M. Garsia, Stanford University, and Mr. R. A. Kunze, University of Chicago, have been appointed C. L. E. Moore Instructors; Mr. D. J. Newman, AVCO Corporation, has been appointed Instructor; Professor Armand Borel, Institute for Advanced Study, will be Visiting Professor during the spring semester; Professors Lennert Carleson, Mathematical Institute of the University of Uppsala, and S. S. Chern, University of Chicago, are Visiting Professors during the fall semester; Dr. M. S. Longuet-Higgins, National Institute of Oceanography, Wormley, England, is Visiting Professor during the academic year 1957–

58; Assistant Professor Jürgen Moser, New York University, is Visiting Assistant Professor during the academic year; Assistant Professor Barrett O'Neill, University of California at Los Angeles, is guest of the Department during the fall semester and will be Visiting Assistant Professor during the spring semester; Associate Professors Warren Ambrose and G. W. Whitehead have been promoted to Professors; Assistant Professors N. C. Ankeny and J. F. Nash have been promoted to Associate Professors; Associate Professor Kenkichi Iwasawa, who has been awarded a Guggenheim Fellowship, is on leave of absence at the Institute for Advanced Study; Professor H. P. McKean, Jr. has been awarded a Fulbright Grant and a National Science Foundation Post-doctoral Fellowship and is on leave at the University of Kyoto, Japan.

North Carolina State College of Agriculture and Engineering: Professor J. W. Cell has been appointed Chairman of the Department of Mathematics; Professor H. A. Fisher, former Chairman of the Department, has retired.

Ripon College: Associate Professor E. G. H. Comfort has been promoted to Professor; Mr. C. W. Larson has been promoted to Assistant Professor.

Wesleyan University: Dr. C. S. Coleman has been promoted to Assistant Professor; Dr. Frederic Cunningham, Jr., Lecturer, Bryn Mawr College, has been appointed Assistant Professor; Dr. R. G. Long, University of Washington, and Dr. John McKibben, University of Chicago, have been appointed Instructors; Professor R. A. Rosenbaum has been awarded a National Science Foundation Faculty Fellowship which he will hold during the academic year 1958–59 at Oxford University; Associate Professor Hing Tong is on sabbatical leave during 1957–58; Professor L. B. Williams, Reed College, is Visiting Professor during 1957–58.

Dr. A. G. Anderson, Mathematician, Glass Division Research Laboratory, Pittsburgh Plate Glass Company, Pittsburgh, Pennsylvania, is employed now as Chief Statistician at the General Tire and Rubber Company, Akron, Ohio.

Miss Allene Archer, Teacher, Thomas Jefferson High School, Richmond, Virginia, has been appointed Professor of Mathematics and Mathematics Education at Maryland State Teachers College, Towson.

Associate Professor B. H. Arnold, Oregon State College, has been granted a Fulbright Award and is teaching at Higher Teachers Training College, Baghdad, Iraq, during 1957–58.

Mr. A. F. Bakenhus, Student, University of Houston, has a position as Mathematician-Programmer for Service Bureau Corporation, Houston, Texas.

Mr. C. W. Barnett, Research Programmer, Louisiana State University, has accepted a position as Applied Science Representative with the International Business Machines Corporation, Fort Worth, Texas.

Assistant Professor J. F. Blackburn, U. S. Air Force Academy, has a position as Education Coordinator with the International Business Machines Corporation, Cambridge, Massachusetts.

Dr. E. K. Blum, Naval Ordnance Laboratory, has accepted a position as a member of the technical staff of the Ramo-Wooldridge Corporation, Los Angeles, California.

Associate Professor R. C. Boles, Mercer University, has been appointed Assistant Professor at North Carolina State College.

Assistant Professor Fred Brafman, Southern Illinois University, has been appointed Associate Professor at the University of Oklahoma.

Whitney Visiting Professor Louis Brand, Trinity College, has been appointed M. D. Anderson Professor of Mathematics at the University of Houston for 1957–58.

Mrs. Carolina del Mar Brennan, University of Philippines, is a graduate research mathematician at the University of California Computer Center, Berkeley.

Dr. R. J. Buehler, Project Associate, Naval Research Laboratory, University of Wisconsin, has been appointed Assistant Professor in Statistics at Iowa State College.

Dr. E. L. Buell, Technical Director, Aerial Measurements Laboratory, Northwestern University, has been appointed Professor at Worcester Polytechnic Institute.

Dr. J. O. Carter, Stanford University, has accepted a position as Applied Science Representative with International Business Machines Corporation, San Jose, California.

Dr. D. R. Childs has a position as a scientist for Westinghouse Electric Corporation, Pittsburgh, Pennsylvania.

Assistant Professor H. E. Chrestenson, Whitman College, has been appointed Assistant Professor at Reed College.

Assistant Professor K. L. Cooke, State College of Washington, has been appointed Assistant Professor at Pomona College.

Mr. Walter Cornetz, Instrumentation Engineer, Grumman Aircraft Corporation, Bethpage, New York, is employed now as Principal Flight Test Engineer, Republic Aviation Corporation, Farmingdale, New York.

Professor W. H. H. Cowles, Chairman of the Department of Mathematics, Pratt Institute, has retired.

Dr. R. H. Crowell, Forrestal Research Center, has been appointed Lecturer at Massachusetts Institute of Technology.

Professor D. R. Davis, Chairman of Department of Mathematics, State Teachers College, Montclair, New Jersey, has been appointed Chairman of the Department of Mathematics of East Carolina College.

Mr. A. C. deWilde, Detroit Edison Company, has accepted a position as Senior Analyst with General Motors, Detroit, Michigan.

Mr. W. C. Dixon has accepted a position as Engineer with the Analysis and Programming Section, BIZMAC Engineering, Radio Corporation of America, New Jersey.

Mr. Carlos Fallon, Systems Engineer, Vitro Corporation of America, is now Chief Mechanical Design Engineer with Nems-Clarke, Silver Spring, Maryland.

Dr. G. E. Forsythe, Research Mathematician, Numerical Analysis Research, University of California at Los Angeles, has been appointed Professor at Stanford University.

Dr. J. W. Frick has been appointed Operations Analyst at Technical Operations, Inc., Monterey, California.

Associate Professor R. L. Garrett, Middle Georgia College, has been promoted to Professor.

Dr. Betty J. Gassner, Graduate Student, New York University, is employed as a research analyst at Remington Rand, New York, New York.

Mr. J. H. Gissel, Instructor, General Motors Institute, is a test engineer for Division of General Dynamics, Consolidated-Vultee Aircraft, San Diego, California.

Mr. B. T. Goldbeck, Jr., of the University of Oklahoma has been appointed to an assistant professorship at Texas Christian University.

Mr. M. L. Goldwater, Research Engineer, J. B. Rea, Santa Monica, California, is a senior engineer for Litton Industries, Beverly Hills, California.

Dr. Ulf Grenander, University of Stockholm, Sweden, has been appointed Professor of Mathematical Statistics and Probability in the Division of Applied Mathematics, Brown University.

Dr. R. M. Gundersen has been appointed Assistant Professor at Illinois Institute of Technology.

Dr. C. J. A. Halberg, Jr., University of California, Riverside, has been promoted to Assistant Professor.

Dr. Carl Hammer of UNIVAC European Computing Center has accepted a position with the Remington Rand International Corporation, New York, New York.

Mr. J. W. Haynes, Jr., Graduate Student, University of California, Berkeley, has a position as a mechanical engineer at Pearl Harbor Navy Shipyard, Honolulu, Hawaii.

Dr. R. T. Herbst, Associate Director, Mathematical Sciences Division, Office of Naval Research, U. S. Army, Durham, North Carolina, is now a member of the technical staff of Bell Telephone Laboratories, Winston-Salem, North Carolina.

Professor P. G. Hodge, Jr., Polytechnic Institute of Brooklyn, has been appointed Professor of Mechanics at Illinois Institute of Technology.

Mr. R. B. Jackson, Jr., of Duke University has been appointed to an assistant professorship at Davidson College.

Mr. W. H. Jones, Mathematician, Department of Defense, Washington, D. C., is now Associate Social Scientist in System Development Corporation, Santa Monica, California.

Professor H. K. Justice, University of Cincinnati, has been appointed Dean of the College of Engineering.

Dr. May R. Kinsolving, Harpur College, has been appointed Instructor at Cornell University.

Mr. A. G. Konheim, Student, Polytechnic Institute of Brooklyn, has been awarded a General Electric Honors Program Fellowship and is studying at Cornell University.

Professor Harry Langman, Chairman of the Department of Mathematics, Detroit Institute of Technology, has been appointed Professor and Chairman of the Department of Mathematics, Ohio Northern University, and Lecturer, University of Pittsburgh, Lima, Ohio.

Dr. Marguerite Lehr, Bryn Mawr College, is on sabbatical leave as visiting fellow at Princeton University.

Assistant Professor L. R. Lewis, State Teachers College, Mankato, Minnesota, is employed as a mathematician with Remington-Rand UNIVAC, St. Paul, Minnesota.

Associate Professor Viktors Linis, University of Ottawa, has been appointed Chairman of the Department of Mathematics.

Professor L. L. Lowenstein, Kent State University, has been appointed Professor at Arizona State College, Tempe.

Professor W. G. Madow, University of Illinois, has been appointed Consulting Professor at Stanford University and Senior Research Statistician at Stanford Research Institute.

Dr. M. E. Mahowald, U. S. Marine Corps, has accepted a position as a mathematician with the General Electric Company, Cincinnati, Ohio.

Dr. M. J. Mansfield, Purdue University, has been appointed Assistant Professor at Washington and Jefferson College.

Dr. J. R. Mayor, Director, Science Teaching Improvement Program, American Association for the Advancement of Science, has been appointed to the newly established position of Director of Education of the A.A.A.S.

Mr. J. S. McNair has been appointed Associate Professor, Teachers College at Plattsburgh, State University of New York.

Assistant Professor B. C. Meyer, University of Arizona, has been appointed Assistant Professor at the University of Colorado.

Professor Emeritus W. L. Miser, Vanderbilt University, recently at McKendree College, has been appointed Professor at Ohio Northern University.

Assistant Professor Irene P. Monahan, Keuka College, has been promoted to Associate Professor and Head of the Department of Mathematics.

Mr. H. G. Moore, Graduate Assistant, University of Utah, has been appointed Instructor at Purdue University.

Associate Professor L. T. Moore of Brooklyn College has retired.

Associate Professor Irene Nolan, West Virginia Institute of Technology, has been appointed Assistant Professor at Tennessee Polytechnic Institute.

Assistant Professor C. S. Ogilvy, Hamilton College, has been promoted to Associate Professor.

Mr. Jack Padgett, Armstrong College of Savannah, has been appointed Registrar of the College.

Associate Professor T. K. Pan, University of Oklahoma, has been promoted to Professor.

Assistant Professor W. H. Peirce, Michigan State University, has a position as Mathematical Analyst with the Electric Boat Division, General Dynamics Corporation, Groton, Connecticut.

Mr. R. J. Pipino, Instructor, Western Reserve University, is Research Staff Assistant at Johns Hopkins University.

Associate Professor R. M. Redheffer, University of California at Los Angeles, was on a sabbatical leave during 1956–57 and spent the fall semester as a National Science Foundation fellow in Göttingen, Germany and the spring semester as a Fulbright Research Scholar in Vienna, Austria.

Mr. A. T. Rice, Test Calculations Liaison Specialist, Investigations Section, General Electric Company, Lynn, Massachusetts, is now a member of the technical staff, Ramo-Wooldridge Corporation, Los Angeles, California.

Professor E. K. Ritter, Director, Rich Electronic Computer Center, Georgia Institute of Technology, has accepted a position as Manager, Mathematical Analysis Department, Lockheed Aircraft Corporation, Marietta, Georgia.

Dr. Azriel Rosenfeld, Junior Physicist, Fairchild Controls Corporation, New York, New York, has a position as Senior Design Engineer, Ford Instrument Company, Long Island City, New York.

Associate Professor Diran Sarafyan, Lamar State College of Technology, has been appointed Assistant Professor at the University of Florida.

Assistant Professor Alice T. Schafer, Connecticut College, has been promoted to Associate Professor.

Mrs. Martha M. Schneider, Student, University of Kentucky, is now a computer programmer for the General Electric Company, Schenectady, New York.

Mr. R. C. Scott, Lehigh University, has been appointed Instructor at Worcester Polytechnic Institute.

Associate Professor C. F. Sebesta, Duquesne University, has been appointed Head of the Department of Mathematics.

Professor Virginia Shuford, Rinehardt College, has been appointed Instructor at Lenoir Rhyne College.

Professor H. W. Smith, Oklahoma Agricultural and Mechanical College, has retired from this position and has been appointed Professor at the University of South Carolina.

Mr. P. H. Thrower, University of Texas, has accepted a position as Applied Science Representative with the International Business Machines Corporation, Dallas, Texas.

Mr. Richard Tukovits, Student, Hunter College, is now an analyst at Babcock and Wilcox, New York, New York.

Dr. H. S. Valk, Research Assistant, Washington University, has been appointed Assistant Professor of Physics at the University of Oregon.

Dr. E. T. Welmers, Chief of Dynamics, Bell Aircraft Corporation, Niagara Falls, New York, has been appointed Director of the Lawrence D. Bell Research Center, Bell Aircraft Corporation.

Associate Professor Albert Wilansky, Lehigh University, has been promoted to Professor.

Assistant Professor Frances M. Wright, Harpur College, has been promoted to Associate Professor.

Mr. Burton A. Yale is a junior mathematician for Cornell Aeronautical Laboratories, Buffalo, New York.

Mr. C. P. Cebulla, Instructor, Hermantown High School, Duluth, Minnesota, died on October 21, 1956.

Mr. B. A. Chiappinelli of Canning, Sisson, and Associates, Los Angeles, California, died on December 20, 1956.

Associate Professor Emeritus A. A. Hatch, Lafayette College, died on May 7, 1957. He was a member of the Association for thirty-four years.

Assistant Professor D. K. Kazarinoff, University of Michigan, died on February 9, 1957.

Professor Emeritus Konrad Knopp of the Eberhard Karls University of Tübingen, Germany, died on April 30, 1957.

Assistant Professor A. G. Makarov, Rutgers University, died on March 25, 1957. He was a member of the Association for sixteen years.

Associate Professor Emeritus V. C. Poor, University of Michigan, died on March 14, 1957.

Mr. W. E. Voronovich, Lewis Machine Company, Cleveland, Ohio, died on February 18, 1957.

Professor R. B. Wildermuth, Capital University, died on September 28, 1956. He was a member of the Association for forty years.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW SECTIONAL GOVERNORS OF THE ASSOCIATION

The following have been elected Governors of the Association for a three-year term beginning July 1, 1957 by a mail vote of the membership of the Association in the Sections indicated:

Allegheny Mountain	J. C. Knipp, University of Pittsburgh
Indiana	Lamberto Cesari, Purdue University
Kentucky	R. S. Park, Eastern Kentucky State College
Metropolitan New York	Jewell H. Bushey, Hunter College
Nebraska	W. G. Leavitt, University of Nebraska
Northern California	George Pólya, Stanford University
Oklahoma	W. N. Huff, University of Oklahoma
Rocky Mountain	C. R. Wylie, Jr., University of Utah
Wisconsin	H. P. Evans, University of Wisconsin

As usual, a high percentage of votes was cast in the elections of sectional governors. In the case of four sections, votes were received from more than 50% of the membership. In the Oklahoma Section, the percentage of votes cast was 58%. In the Kentucky, Rocky Mountain, and Wisconsin Sections it was 54%.

H. M. GEHMAN, *Secretary-Treasurer*

THE 1957 COMBINED MEMBERSHIP LIST

The Mathematical Association of America joins with the American Mathematical Society and the Society for Industrial and Applied Mathematics each year in publishing a Combined Membership List. Each member of the Association will receive a copy of the

1957 List as one of the privileges of membership. It is expected that the 1957 List will be ready for distribution in December 1957.

Prompt notice should be given to the office of the Association of all changes in position, rank, and address which have not been reported previously. Any errors in the 1956 List should also be reported. The final date for receipt of changes is October 15.

HARRY M. GEHMAN, *Secretary-Treasurer*

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following 131 persons have been elected to membership by the Board of Governors on applications duly certified.

JONATHAN L. ALPERIN, Student, Harvard University.

JUDITH AMES, Student, Vassar College.

MARY ASTERITA, Res. Asst., Linde Air Products Co., New York, N. Y.

DUANE W. BAILEY, Student, State College of Washington.

LOUIS A. P. BALÁZS, Student, University of California.

RALPH BARGEN, B.S. (Bethel) Grad. Student, University of Wichita.

NYLES N. BARNERT, Student, Haverford College.

ROBERT L. BARROWS, B.S. (U.C.L.A.) L.C.D.R., U. S. Navy, Coronado, Calif.

MARSHALL C. BELL, M.A. (North Carolina) Asso. Professor, Clemson College.

DUDLEY J. BERTRAND, Student, Southwestern Louisiana Institute.

ALBERT A. BLANK, Ph.D. (N.Y.U.) Asst. Professor, University of Tennessee.

LOIS A. BLUM, A.B. (Michigan) Teaching Fellow, University of Michigan.

VOLODYMYR BOHUN-CHUDYNIV, Professor & Chm., Department of Mathematics, Atlanta University.

JOHN A. BOND, JR., Student, Texas Technological College.

JOHN K. BOOTH, Student, Muskingum College.

JULIEN L. BORDEN, B.A. (U.C.L.A.) Los Angeles, California.

MRS. LUCILLE C. BORGMAN, M.A. (Michigan) Asst. Professor, Detroit Institute of Technology.

CLAUDE BOUCHER, M.S. (Montreal) Asst. Professor, University of Sherbrooke, Canada.

DAVID R. BRILLINGER, Student, University of Toronto.

MAURICE BRISEBOIS, B.A. (Montreal) Grad. Student, University of Montreal.

JOHN B. BRONZAN, Student, Stanford University.

MRS. EVA T. BROWDER, B.S. (M.I.T.) Teacher, Hillhouse High School, New Haven, Conn.

JOSEPH O. CARTER, Ph.D. (Stanford) Applied Science Rep., I.B.M. Corp., San Jose, Calif.

DONALD A. CELARIER, Student, Northwestern University.

FRANK M. CHOLEWINSKI, Student, Alabama Polytechnic Institute.

INGE F. CHRISTENSEN, M.A. (Catholic) Grad. Asst., Catholic University of America.

RONALD A. CHRISTENSEN, Student, Iowa State College.

CHARLES A. CHURCH, JR., Student, Virginia Polytechnic Institute.

MOTHER MARY CLETUS, S.H.C.J., M.A. (Pennsylvania) Head, Department of Mathematics, Rosemont College.

HASKELL COHEN, Ph.D. (Tulane) Asst. Professor, Louisiana State University.

WILLIAM L. CONGLETON, Student, Harvard University.

VINCENT F. COSTANZA, M.A. (San Jose S.C.) Teacher, San Jose Junior College.

ANNE T. COYLE, B.A. (St. Rose) Teacher, New York Mills High School, N. Y.

GORDON F. DALSIN, M.A. (Washington) Professor, Canadian Services College, Royal Roads, B. C.

CATHERINE J. DAVIS, B.A. (Spelman) Grad. Student, Atlanta University.

JOHN S. DENTON, JR., Student, University of Michigan.

ANDRIES C. DEWILDE, M.S.M.E., M.S.E.E. (Technical U., Delft) Sr. Analyst, General Motors Technical Center, Detroit, Mich.; Asst. Prof. of Mech. & Electrical Engg., Detroit Institute of Technology.

GEORGE E. DUNNE, B.E. (Liverpool) Site Engr., Bell Telephone Co. of Canada, Montreal, P. Q.

HARRISON A. D. DUNSWORTH, M.A. (Southern Methodist) Head, Department of Mathematics, Arlington State College.

ROBERT E. EDIE, Student, University of Buffalo.

MERRITT ELMORE, M.A. (California) Teacher, San Jose Junior College.

JEREMIAH P. FARRELL, Student, University of Nebraska.

NORMAN V. FELLERS, JR., B.A. (Hawaii) Grad. Student, American Institute for Foreign Trade.

JOHN F. FIRKINS, JR., Student, St. Martin's College.

GEORGE R. FLOWERS, M.S. (Emory) Decatur, Georgia.

JEAN FORTIER, B.A. (St. Thérése) Grad. Student, University of Montreal.

DON W. FOSTER, B.S.E.E. (Washington) Engr., Weldit Tank & Steel Co., Bellingham, Wash.

WILLIAM G. FRANZEN, M.S. (Miami) Asst. Professor, Aquinas College.

JOSEPH J. GARSIDE, A.B. (Duke) Test Engr., Convair-Astronautics, San Diego, Calif.

H. PHILIP GEORGE, Student, University of Rochester.

BERNARD GILBERT, M.S. (N.Y.U.) Res. Aide, New York University, College of Engineering.

LOUIS J. GRANATO, Student, Rutgers College of South Jersey.

HENRY C. GRIFFIN, Student, Davidson College.

BROTHER FRANK R. GUTTING, S.M., M.A. (St. Louis) Instr., St. Mary's University.

JOHN R. HANNE, Student, Dartmouth College.

PAUL V. HANSEN, JR., M.S. (Northwestern) Instr. in Chemistry, Dana College.

JOHN M. HARRINGTON, Ph.M. (Wisconsin) Professor and Head, Department of Mathematics, Michigan College of Mining and Technology.

FLOYD K. HEIDEN, Student, St. Mary's University.

GABRIEL M. HELLMAN, M.S. (N.Y.U.) Actuary, Pension Planning Co., New York, N. Y.

NORMAN P. HERNBERG, Student, Lebanon Valley College.

MORTON A. HIRSCHBERG, Student, University of California at Los Angeles.

EDWARD B. HOFF, B.S. (Arizona) Grad. Asst., University of Arizona.

WILLIAM W. HOKMAN, M.A. (West Virginia) Instr., West Virginia University.

PAUL T. HOLMES, Student, State College of Washington.

RICHARD E. HUGHS, Student, University of Rochester.

JACK L. HURSCH, JR., M.A. (Denver) Teaching Asst., University of California.

MRS. FLORENCE D. JACOBSON, S.M. (Chicago) Instr., Albertus Magnus College.

ARNO JAEGER, Dr.rer.nat. (Göttingen) Asso. Professor, University of Cincinnati.

ERNEST H. KANNING, III, B.A. (Valparaiso) Grad. Asst., Ohio State University.

RICHARD H. KERR, M.S. (Missouri School of Mines) Asst. Professor, Missouri School of Mines.

DONALD A. KING, M.S. (Purdue) Asst. Professor, Clemson College.

JOAN KIRKHAM, A.B. (Kansas) Res. Math., Denver Research Institute, University of Denver.

JOHN C. LEAVY, JR., B.A. (N.Y.U.) Grad. Asst., Rutgers University.

MILTON LEVY, B.A. (Cornell) Numerical Analyst, Electro-Mechanical Laboratories, White Sands Proving Ground, N. Mex.

LEON M. LEWANDOWSKI, Student, University of Buffalo; Math.-Engr., Sylvania Electric Corp., Buffalo, N. Y.

JOHN A. LUTTS, S.J., Student, Spring Hill College.

E. MYRTICE LYNCH, M.A. (Columbia) Chm., Department of Mathematics, O'Keefe High School, Atlanta, Ga.

VICTOR M. MANJARREZ, S.J., Student, Spring Hill College.

T. EDWARD MARTIN, Denver, Colorado.

GEORGE M. MATOUS, Student, St. Mary's University.

ETHELYNE L. MCBEE, M.A. (Columbia) Statistician, Florida Industrial Commission, Tallahassee.

JOSEPH H. MCBETH, Student, Baylor University.

JOHN P. MCCABE, Student, Manhattan College.

CUTHBERT L. MCCARTY, JR., Student, Georgia Institute of Technology.

RICHARD F. MCDERMOT, Student, Carnegie Institute of Technology.

BERNARD J. MCGOVERN, A.B. (Providence) Math., Combustion Engineering, Windsor, Conn.

JAMES H. MCINTYRE, M.A. (North Carolina) Asst. Professor, The Citadel.

JAMES J. MCMAHON, Student, University of Detroit.

CHARLES J. MELUCK, Machine Designer, E. Hofmann, Chesterland, Ohio.

JOHN H. MILES, B.S. in E.E. (Texas A. & M.C.) Engr., Atlanta, Ga.

HENRY C. MILLER, JR., M.S. (Chicago) Instr., University of Alabama.

JOHN C. MILLER, Student, Oberlin College.

GEORGE W. MORRIS, JR., M.S. (Oklahoma A. & M.C.) Math., U. S. Naval Air Missiles Test Center, Point Mugu, Calif.

STANLEY P. MORRIS, Student, McGill University.

FRANCOIS MUNIER, B.S. (Montreal) Asst. Professor, University of Montreal.

JEFFYE V. NORWOOD, Student, New Mexico College of Agriculture and Mechanic Arts.

JOHN O. OTTERNESS, Student, St. Olaf College.

DONALD J. PERSICO, Student, University of Buffalo.

JERRY L. PIETENPOL, Student, Davidson College.

EDWARD R. PINCUS, Student, Brown University.

LARRY H. POTTER, M.A. (Florida) Asst. Professor, Memphis State College.

WILLIAM J. PRICE, Student, Long Beach State College.

JOSEPH T. RATCHFORD, Student, Davidson College.

OLIVER W. REESE, B.S. (Ohio S.U.) Math., National Advisory Committee on Aeronautics, Cleveland, Ohio.

KENNETH R. RUNYAN, Student, William Jewell College.

JUDSON SANDERSON, JR., Ph.D. (Illinois) Asso. Professor, University of Redlands.

ALBERT SCHILD, Ph.D. (Pennsylvania) Asst. Professor, Temple University.

WALTER A. SCHNEIDER, A. O. Smith Corp., Milwaukee, Wis.

JOHN B. SCOTT, Student, University of Arizona.

MARGARET M. SCULLY, Ed.M. (Boston U.) Teacher, Emma Willard School, Troy, N. Y.

ALFRED M. SHOLANDER, Student, Seton Hall University.

LEWIS J. SIMONOFF, M.A. (Syracuse) Grad. Student, University of California.

SISTER MARY DOLORIAN, M.A. (Catholic) Instr., Briar Cliff College.

SISTER MARY LAETITIA HILL, Ph.D. (Catholic) Teacher, Our Lady of the Lake College.

OSSIE M. SMITH, A.B. (Spelman) Grad. Student, Atlanta University.

NORMAN P. STEIN, M.S. (Chicago) Chm., Department of Mathematics, Wilson Junior College.

DAN W. STODDARD, M.S. (Utah S.A.C.) Asst. Professor, Brigham Young University.

GERALD S. STOLLER, Student, Polytechnic Institute of Brooklyn.

CLAUDE E. STOUT, M.A. (Wisconsin) Chm., Department of Mathematics and Engineering Mechanics, General Motors Institute Flint, Mich.

CHARLES J. STUTH, M.Ed. (East Texas S.T.C.) Instr., East Texas State Teachers College.

THEODORE M. SZYDLOWSKI, M.S. (Northwestern) Asst. Professor, St. Mary's University.

MELVIN C. THORNTON, Student, University of Nebraska.

PAUL H. THROWER, M.A. (Texas) Applied Science Rep., I.B.M. Corp., Dallas, Texas.

RICHARD L. USCHOLD, M.S. (Notre Dame) Instr., Canisius College.

DAVID C. WEISER, Student, Brooklyn College.

MICHAEL WELITSCHKOFF, Dip. Math. (Belgrad) Computer, Republic Aviation Corp., Farmingdale, N. Y.

WARREN S. WENGER, Student, Lebanon Valley College.

JEAN L. WHITE, Student, University of Maine.

ROSCOE WHITE, Student, University of Minnesota.

LEROY M. WILLSON, M.A. (Georgia) Asso. Professor, Georgia State College of Business Administration.

C. DUANE ZIMMERMAN, Student, Emmanuel Missionary College.

THE MARCH MEETING OF THE MICHIGAN SECTION

The annual meeting of the Michigan Section of the Mathematical Association of America was held on March 23, 1957 at Wayne State University, Detroit, Michigan in conjunction with the meeting of the Michigan Academy of Science, Arts and Letters. Professor R. S. Pate of Eastern Michigan College presided at both the morning and afternoon meetings and at the luncheon-business meeting. A total of 81 persons attended the meeting including 63 members of the Association.

A nominating committee consisting of Professors C. C. Richtmeyer, Chairman, P. S. Jones and D. C. Morrow proposed Professor A. E. Lampen of Hope College for Chairman and Professor F. A. Beeler of Western Michigan University for Secretary-Treasurer. The slate was elected unanimously.

The Section discussed the possibility of conducting a mathematics contest for high school students. It was reported that Michigan State University had been giving some thought to such a contest. Also the Industrial Mathematics Society of Detroit had been making plans for a contest for the greater Detroit area. It seemed expedient to appoint a committee to study this proposal with the idea of unity of efforts for these groups. The new Chairman, Professor Lampen, appointed a committee consisting of Professors R. H. Oehmke of Michigan State University, A. W. Jacobson of Wayne State University and A. J. Lohwater of the University of Michigan to study this possibility of a mathematics contest for high school students in Michigan and report its findings and recommendations at the next meeting.

A panel discussion was held on the subject of "Recent Undergraduate Curricular Changes in Michigan." The panel consisted of the following: Professor J. S. Frame, Michigan State University; Professor E. W. Goings, Eastern Michigan College; Professor G. E. Hay, University of Michigan; Professor Felix Haas, Wayne State University; Mr. J. S. Griffin, University of Michigan; Professor L. E. Mehlenbacher, University of Detroit; Dr. E. S. Northam, Wayne State University and Professor R. C. Seber, Western Michigan University.

Trends evident from the discussion were the following: to challenge students more effectively; to avoid repetition of high school work; to cooperate more closely with other departments; to introduce calculus, vector analysis, matrices, *etc.* earlier; to increase the rigor in select sections; and to introduce courses cutting across, modernizing, and extending the conventional compartment-type courses. The reaction of students and faculty to such changes has been good.

The following papers were presented:

1. *Topological semi-groups of triangulable spaces*, by Professor S. T. Hu, Wayne State University.

Let X be a topological semi-group defined on a compact connected locally triangulable space and having a two-sided unit u. If u is a homotopically stabil point of the space X [Borsuk, *Fund. Math.*, vol. 39, 1952, p. 159], then it is proved in this paper that X is a topological group. Hence, if X has no homotopically labil point, then X must be a topological group. This generalizes Wallace's theorem that *a topological semi-group with a two-sided unit defined on a compact connected manifold must be a topological group*, [*Bull. Amer. Math. Soc.*, vol. 61, 1955, p. 96]. A special case of this generalized result is that a topological semi-group with a two-sided unit defined on a compact connected linear graph with no end point must be a topological group and hence it is the circle group.

2. *Pararegular polygons*, by Professor B. M. Stewart, Michigan State University.

Let a regular polygon of order n be constructed outwardly (or inwardly) on each edge of a given closed polygon C. Then a necessary and sufficient condition that the centers of these regular

polygons form a regular polygon of order n is that C be a *pararegular* polygon of order n, *i.e.*, that C be obtained by parallel projection from a regular polygon of order n.

3. *A simple proof of the Bernstein equivalence theorem*, by Professor A. H. Copeland, Sr., University of Michigan.

The Bernstein equivalence theorem states that *if each of two sets is mapped into a subset of the other by one of two transformations* (1-1 *correspondences*) *then the sets are in* 1-1 *correspondence with each other*. This proof consists in using the given transformations together with two additional sets to produce the latter correspondence. The four sets and two transformations satisfy two conditions which are treated as though they were algebraic equations and solved formally for the additional sets. The solution is shown to produce the desired correspondence.

4. *Approximating a circular segment by use of Diophantine equations*, by Professor J. S. Frame, Michigan State University.

A circular segment with height $h=1-\cos\theta<1$, arc length $2\theta<\pi$, chord length $2\sin\theta$, and area $S=\theta-\sin\theta\cos\theta$, is cut from a unit circle. The area of the inscribed isosceles triangle of height h is called $T=\sin\theta(1-\cos\theta)$. The segment-triangle ratio $S/T=G(h)$ and the arc-chord ratio $\theta/\sin\theta=F(h)$, are related by the equation $G-1=(F-1)/h$. An approximation to $F(h)$ of the form $f(h)=a/(1-bh)+(1-a)/(1-ch)$ is found by Diophantine equations yielding $a=5/(5+u^2)$, $3b=1+u/5$, $3c=1-1/u$, whose series agrees to three terms with the series for $F(h)$. This gives the close rational approximation $311/198=1.5707$ for $\pi/2$ at $h=1$ when $u=7/4$. Thus approximations with errors about $h^3/2100$ and $h^2/2100$ for small h and with maximum errors for $h<1$ of 0.00017 and 0.00023 respectively are found as follows:

$$\theta/\sin\theta \doteq (420-109h)/(20-h)(21-3h), \qquad h=1-\cos\theta<1,$$

$$\text{Segment/Triangle} \doteq 1+(140-27h)/(20-9h)(21-3h).$$

5. *On certain conformal maps in three space*, by Professor M. O. Reade, University of Michigan. (Presented by title.)

In this note the following theorem is proved. *If* $y=y(x)$ *is a conformal differentiable homeomorphism of* $\|x\|<1$ *onto* $\|y\|<1$, *with nonvanishing Jacobian, and if* $y(0)=0$, *then* $y(x)$ *is a rotation*. The method is an elementary one depending upon properties of subharmonic functions.

6. *The set of Lusin points of an analytic function*, by Professor George Piranian, University of Michigan and Professor A. L. Shields, University of Michigan, presented by Professor Shields.

Let $f(z)$ be regular in $|z|<1$; a point $e^{i\theta}$ is called a Lusin point of f provided f maps every disk which is internally tangent to the unit circle $e^{i\theta}$ onto a Riemann surface of infinite area. THEOREM: *The set of Lusin points of* f *is a set of type* G_δ; *conversely, if* E *is a set of type* G_δ *on the unit circle, it coincides with the set of Lusin points of some function regular in* $|z|<1$.

7. *The discriminant function applied to methods of preparing fish*, by Professor W. D. Baten, Michigan State University.

This paper shows how discriminant functions may be used to distinguish methods of preparing smoked fish where four observations on flavor, aroma, moisture and texture were made on each fish by a panel of judges. These discriminant functions enable one to determine which of the above was the most important measurement, the next of importance, *etc.* The paper develops the relationship between discriminant functions and multiple regressions.

8. *A procedure for clique detection using a group matrix*, by Professor Frank Harary, University of Michigan and Professor I. C. Ross, University of Michigan.

One of the unsolved problems in the analysis of sociometric data is the complete identification of the cliques in a group of persons. Using the concept of a unicliqual person in a group and a

procedure of successive partitioning, we obtain a matric algorithm for detecting all the cliques in any group. This is equivalent to determining all the maximal complete subgroups in any directed graph using the adjacency matrix.

9. *Ambiguous points of a function defined inside a sphere*, by Professor George Piranian, University of Michigan.

There exists a function $f(x, y, z)$, continuous in the sphere $x^2+y^2+z^2<1$ and possessing the following property: for every real number A and every point P on the surface $x^2+y^2+z^2=1$, the sphere contains a rectifiable path $C(A, P)$ from the origin to P such that $f(x, y, z)\to A$ as $(x, y, z)\to P$ along $C(A, P)$.

F. A. BEELER, *Secretary*

THE APRIL MEETING OF THE IOWA SECTION

The forty-fourth annual meeting of the Iowa Section of the Mathematical Association of America was held at Iowa State Teachers College, Cedar Falls, Iowa, April 26–27, 1957. Professor F. W. Lott, Chairman of the Section, presided at the April 26 afternoon session and Professor A. H. Blue, Vice-Chairman, presided at the April 27 morning session. Total attendance was 63, including 37 members of the Association.

Routine business was considered and a committee report was made on arrangements for a joint meeting with the Iowa Association of Mathematics Teachers.

It was voted that the Chairman should appoint a committee to study the problems of examinations (contests), to meet with other groups interested in sponsoring the examination, and to report to the Executive Committee of the Section for a decision regarding the matter. It was also voted that the group express itself as being in favor of such a contest.

The following officers were elected: Chairman, Professor A. H. Blue, Cornell College; Vice-Chairman, Professor A. T. Craig, The State University of Iowa; Secretary-Treasurer, Professor E. L. Canfield, Drake University.

The following papers completed the program:

1. *Report of the committee on cooperation among college and high school teachers of mathematics*, by Professor O. C. Kreider, Iowa State College, Professor M. F. Smiley, State University of Iowa and Professor I. H. Brune, Chairman of Committee, Iowa State Teachers College, presented by Professor Brune.

Members of the Iowa Section of the Mathematical Association of America met with the Iowa Association of Mathematics Teachers on November 2, 1956. Some high school teachers met with the Iowa Section on April 26, 1957. Members of both organizations have been invited to a conference on mathematics at the State University of Iowa in October 1957.

The committee recommended that members of the Iowa Section: (1) maintain membership in the IAMT; (2) meet whenever possible with high school teachers; (3) contribute items for the IAMT Newsletter; (4) speak at meetings of high school teachers; (5) sponsor meetings for both groups of teachers.

2. *A trick in solving a class of boundary value problems*, by Professor Don Kirkham, Iowa State College.

In a class of boundary value problems it is found that the solution $f(x)$ reduces to a series $\sum A_m \sin(m\pi x/b)$, $m=1, 2, \cdots, \infty$, defined for $0<x<a$, $a<b$, b being the length of the boundary, and the coefficients A_m to be determined. In this same type of problem the boundary condition is satisfied independently of the series for $a<x<b$ and therefore one may use the trick of setting the series equal to itself for $a<x<b$ in the form $\sum A_m \sin(m\pi x/b)=\sum A_p \sin(p\pi x/b)$, $p=1, 2, \cdots, N$, $N\to\infty$. Then A_m may be found from a series of N simultaneous equations which arise. The trick works also for Bessel-Fourier series *etc.*

3. *Some "solutions" of inconsistent linear systems*, by Professor C. E. Langenhop, Iowa State College.

Let A be an n by r matrix, $n>r$, and let c and x be n and r dimensional column vectors, respectively. The system (1) $Ax=c$ is generally inconsistent. The least squares solution, $\hat{x}$, must satisfy $A'Ax=A'c$ (B'=transpose of B), which is always consistent. Let (2) $A_i x=c_i$ denote a system of r equations from (1) and denote the solution of (2) by $x^{(i)}$ if it exists. Then if A has rank r it was shown that $\hat{x}=|A'A|^{-1}\sum_i |A_i| x^{(i)}$, where $|B|$ =determinant of B and the sum is over all distinct systems (2), the summand being interpreted as a zero if $|A_i|=0$. Various extensions of this relation were also presented.

4. *Filters and equivalent nets*, by Professor M. F. Smiley, State University of Iowa.

This presentation was an attempt to convey the essential technical idea in pedagogical terms by proposing the question as to which of the nets, Basic Mathematical Concepts or Basic Mathematical Skills, is the finer one to catch college freshman.

5. *The cardinal number of residual sets*, by Professor U. R. Kodres, Iowa State College, introduced by the Chairman.

An elementary construction is used to prove the known result that every residual set has the cardinal number of the continuum.

6. *Remarks on the convolution of real functions in Laplace transform theory*, by Professor C. G. Maple and Professor Bernard Vinograde, Iowa State College, presented by Professor Vinograde.

A real function is of class T if it is: (1) of exponential order and (2) sectionally continuous except for a possible infinite discontinuity of order $t^{-\alpha}$, $(0<\alpha<1)$, at $t=0$. It is shown that functions of class T possess absolutely convergent Laplace transforms and furthermore the convolution of two functions of class T is also of class T.

7. *Mathematical training for the exceptional student*, by Professor Fred Robertson, Iowa State College.

The author discusses the program in mathematics for these students as conducted at the Iowa State College since 1946. Tables showing accomplishments in some lines are given. Some results and evaluations of the program are obtained.

8. *An analysis of error in the learning of algebra*, by Professor Vivian Strand and Professor D. A. Yos, Burlington High School and College, presented by Professor Yos, introduced by the Chairman.

The errors which learners of mathematics make are a clue to their thoughts. Some of the more common of these errors have been listed and described. The erratic pattern of error, common errors in addition and division, errors in the addition of unlike fractions, and the compounding of error in the solution of simultaneous equations are treated. The psychology of fear is considered to be a very important factor in initiating these errors.

9. *A remark on a certain sufficient statistic*, by Professor A. T. Craig, State University of Iowa.

In a regular case of estimation let X denote a continuous type random variable having probability density function $f(x;\theta)=\exp[p(\theta)K(x)+S(x)+q(\theta)]$. Let $Y=K(X)$. It is proved that $E(Y)=b+cp(\theta)$, $c>0$, is both necessary and sufficient for Y to be normally distributed. Hence, if $X_1,\cdots,X_n$ denote a random sample of $n>1$ values of X, then $Z=\sum_i K(X_i)$ is a sufficient statistic for θ, and Z will be normally distributed if and only if Y is normally distributed.

10. *On quadratic forms whose distributions are free of the population mean*, by Professor R. V. Hogg, State University of Iowa.

Let $Q_1, Q_2, \cdots, Q_k$ be k nonnegative real symmetric quadratic forms in n random values of a normally distributed variable with mean m. A necessary and sufficient condition that Q_i, $i=1, 2, \cdots, k$, has a distribution free of m is that their sum $Q=\sum_i^k Q_i$ has this property.

11. *A modern approach to elementary analysis*, by Professor H. C. Trimble, Iowa State Teachers College. (By invitation).

While mathematics has grown up in the past one-hundred fifty years, the mathematics taught to high school pupils and college freshmen is seldom more modern than Euler. Experience at Iowa State Teachers College suggests that portions of modern mathematics are teachable. Ninth graders readily assimilate modern ideas of variable, unknown, parameter, relation, function, and the like. College freshmen resist these ideas for a time, and then progress rapidly. Is it true that "mathematics is teachable just because it is good mathematics?"

E. L. CANFIELD, *Secretary*

THE APRIL MEETING OF THE KANSAS SECTION

The forty-second annual meeting of the Kansas Section of the Mathematical Association of America was held at the University of Kansas, Lawrence, Kansas, on April 13, 1957 in conjunction with the annual meeting of the Kansas Association of Teachers of Mathematics. There were 72 persons registered including 40 members of the Association. Professor W. R. Scott, Chairman, presided at the sessions.

The following officers were elected for one year terms: Chairman, Professor L. E. Laird, Kansas State Teachers College, Emporia; Vice-Chairman, Professor P. S. Pretz, St. Benedict's College; Secretary-Treasurer, Miss Helen Kriegsman, Kansas State Teachers College, Pittsburg.

At the business meeting, the section went on record as opposing the sponsorship of high school contests at this time; however, the Executive Committee was requested to investigate the situation and present a report at the 1958 meeting.

The following short papers were presented:

1. *Implicit functions in analog computation*, by Professor P. G. Kirmser, Kansas State College.

The basic components of electronic analog computers are high gain amplifiers and function generators which perform the operations Gx and $f(z, x)$ given (z, x). Using an amplifier and function generator together enables the solution of $f(z, x)=0$, $|x|<K$, $|z|<K$ to be obtained for each x by scanning the (z, x) plane along the line x beginning at $(0, x)$. If z is multiple valued, several scannings must be made from different starting points to insure completeness of solution. A sufficient condition for stability of operation is that $f(z, x)$ given (z, x) is computed more rapidly than Gx. Generalizations lead to the solution of simultaneous implicit equations the form of which is limited only by the practical construction of adequate function generators.

2. *Operations research*, by Professor A. M. Feyerherm, Kansas State College.

The speaker gave a general outline of the methodology used in operations research. The growing interest in the use of mathematical tools and techniques to attack problems whose solutions require knowledge from a variety of disciplines was discussed. Interest in solving such problems carries various implications for future mathematical activities.

3. *Axiomatic algebra for freshmen*, by Professor W. C. Doyle, Rockhurst College.

Postulational algebra for college freshmen with a weak background presents serious pedagogical difficulties. A series of lessons was presented for discussion along with the results of their experimental use in the classroom. The natural numbers are defined in terms of their commutative, distributive, and closure properties. It is then postulated that equations of the form, $a+x=b$ and

$a \cdot x = b$, have solutions for any pair of elements a and b. A series of 23 theorems are proved to show the existence of identity elements, inverses, and their properties. The additive inverse of a is written $\underline{a}$ (instead of $-a$); the multiplicative inverse is written $/a$. The theory is accompanied with a copious supply of manipulative problems.

4. *Simple circuits*, by Dr. R. W. Gibson, Wakeeney, Kansas.

Simple circuits, by electrical analogy, are defined as finite sets of independent (*SPST*) switches, represented by line segments with only endpoints in common (topological graphs); two junctions are terminals. Routes between terminals correspond to terms in the expression by the algebra of sets applied to the switches. Factorization into subcircuits has a corresponding modified factorization of the expression. The factorization into prime subcircuits is unique (parallel and series retained unfactored) and a prime subcircuit has a unique expression. An inverse of a circuit is open instead of closed (switch continuity); factorization applies; a coplanar circuit has a readily constructed inverse.

HELEN KRIEGSMAN, *Secretary*

THE APRIL MEETING OF THE KENTUCKY SECTION

The spring meeting of the Kentucky Section of the Mathematical Association of America was held on April 27, 1957 at Berea College, Berea, Kentucky. Professor T. M. Wright, Berea College, presided at both the morning and afternoon sessions. There were 40 persons in attendance, including 28 members of the Association.

The following officers were elected for one-year terms: Chairman, Professor J. C. Eaves, University of Kentucky; Secretary-Treasurer, Professor V. F. Cowling, University of Kentucky; Traveling Lecturer, Professor W. H. Spragens, Jr., University of Louisville.

By invitation of the Committee, Professor H. D. Brunk of the University of Missouri delivered an hour address at the afternoon session entitled, "Sensitivity Experiments, Stochastic Approximation, and Estimation of Ordered Parameters." An abstract of this address follows:

Sensitivity experiments (yielding "all-or-none" data) were contrasted with ordinary random sampling. Generalizations were illustrated by examples, including one representing a problem in statistical "discrimination." The examples were discussed from the distinct points of view of stochastic approximation and estimation of ordered parameters.

The following papers were presented:

1. *On bounded univalent functions*, by Professor V. F. Cowling, University of Kentucky.

In this paper results analogous to those obtained by Hardy, *Quart. J. Math.* vol. 44, 1913, pp. 147–160, for functions regular and bounded in the unit disk were obtained for that subclass of these functions which are in addition univalent in the unit disk.

2. *Finite difference approximations to partial differential equations*, by Professor W. C. Royster, University of Kentucky.

In solving partial differential equations by numerical methods the topics of convergence and stability are of prime importance. In this paper a discussion of these topics was given for a parabolic differential equation.

3. *Graphs of Chebyshev polynomials*, by Professor W. H. Spragens, University of Louisville.

The Chebyshev polynomials may be represented graphically as plane projections of trigonometric curves drawn on a circular cylinder. Through this representation, many of their properties are readily referred to familiar properties of the trigonometric functions. Among such properties are odd-or even-ness, nature and distribution of zeros and extrema, and orthogonality.

4. *Problems in matrix differential equations*, by Professor T. J. Pignani, University of Kentucky.

This note pointed out the relationship between a linear differential equation and its associated matrix differential equation. Further, bounds for the solution of an interface problem were developed as this problem is expressed in the *J. Elisha Mitchell Sci. Soc.*, vol. 72, 1956, no. 1. Throughout this note open problems were cited.

5. *A paragraph in the topology of algebra*, by Professor J. G. Horne, University of Kentucky.

An application of the Heine-Borel theorem yields the following theorem. *Let f_0 and h_0 be continuous functions over a closed interval I and suppose that for each x in I a neighborhood U and a continuous function h can be found such that $f_0(t) = h(t)h_0(t)$ for all t in U; then there is a continuous function h_1 defined over I such that $f_0(t) = h_1(t)h_0(t)$ for all t in I.*

This result can be formulated and proved in such a way as to generalize to a wide class of ideal structure theorems. The clues needed are these: 1) agree to discuss compactness (the Heine-Borel property) of a set X with respect to a collection T of its subsets, without any concern for whether T is a topology; *e.g.*, if S is a commutative ring and X is a family of its ideals, let T be the collection of subsets $O(f)$ of X which contain the fixed element f of S; 2) agree that the members M of X have the property that if f is in M then there is an e in M such that $fe = f$. The stipulation that X be T-compact yields some interesting results.

6. *Finite projective geometries*, by Professor A. W. Goodman, University of Kentucky.

This talk was an expository talk in which the speaker presented a few of the results contained in the paper of the same title by O. Veblen and W. H. Bussey, *Trans. Amer. Math. Soc.*, vol. 17, 1906, pp. 241–259.

7. *On rearrangements of infinite series*, by Mr. K. E. Stoll, University of Kentucky.

This was an expository talk based on a paper with a similar title by E. Steinitz, *J. Reine Angew. Math.*, vol. 143, 1913, pp. 128–175.

8. *Report on the Kentucky lecture program*, by Professor J. C. Eaves, University of Kentucky.

This paper is a summary of the program carried out by the elected lecturer of the Kentucky Section of the Mathematical Association of America during the two year period 1955–57. The lecturer visited more than one hundred schools during this period and spoke to between 10,000 and 15,000 high school and college students on mathematics and on the need for mathematicians, on the present demand for mathematicians and the various phases of the work that they would perform both in industry and in the teaching field. A series of questions usually asked at these meetings was discussed together with the answers used. It was also brought out that the attitude of many high school mathematics students changed considerably after these visits. It is the belief of the lecturer that more programs of this kind should be set up and especially that the Kentucky Lecture Program should be continued.

V. F. COWLING, *Secretary*

THE APRIL MEETING OF THE NEBRASKA SECTION

The thirty-third annual meeting of the Nebraska Section of the Mathematical Association of America was held on April 26, 1957, at Lincoln, Nebraska, in conjunction with the meetings of the Nebraska Academy of Sciences. Professor J. M. Earl of the University of Omaha presided. There were 52 persons in attendance, including 31 members of the Association.

At the business meeting, the following resolution was passed:

The Nebraska Sections of the Mathematical Association of America and the National

Council of Teachers of Mathematics will jointly sponsor the national mathematics contest within Nebraska, sharing the responsibility of publicizing, administering, and grading the examinations and of making the awards. The problems of carrying out the program as provided by the national organization will be handled by a joint committee of the two sections.

The following officers were elected for 1957–1958: Chairman, Professor Edwin Halfar, University of Nebraska: Vice-Chairman, Professor J. M. Earl, University of Omaha; Secretary-Treasurer, Professor H. M. Cox, University of Nebraska.

The following papers were presented:

1. *Some derived curves*, by Professor H. L. Rice, University of Omaha.

As the point P moves along a curve whose equation is given in polar coordinates, the point N (the intersection of the normal from P and the subnormal) and the point T (the intersection of the tangent at P and the subtangent) trace their respective loci. Let the radii vectors of the points P, N, and T be designated p, R, and r, respectively. Then $R \cdot r = p^2$. This relation leads to some interesting observations regarding inverse points, harmonic properties, and geometric illustrations of indeterminate forms and their limits.

2. *A study on the use of science counselors*, by Mr. M. L. Keedy, University of Nebraska.

Nebraska is one of four states which have received a grant, administered by the American Association for the Advancement of Science, to conduct a study on the use of traveling consultants to high school teachers of mathematics and the sciences. Thirty-nine selected high schools are participating in the Nebraska study. The study shows promise of: (1) focusing the attention of students, teachers, and school administrators on the importance of mathematics and the sciences; (2) raising the level of instruction in these subject-matter areas; and (3) helping to establish better cooperation between schools and colleges.

3. *The mathematics profession*, by Mr. H. W. Becker, Radio Engineering Institute, Omaha, Nebraska.

It is now generally agreed that engineering is a profession. In increasing degree, much of the content of theology, law, medicine, and engineering is mathematical; this implies that mathematics is a profession. Lawyers, doctors, and engineers must be examined and licensed by the state before practising. This legal standing might be advantageous to the growing number of mathematical consultants, available to all mathematicians (in whom, however, the amateur tradition remains strong).

4. *Generalisation of groups*, by Professor Edwin Hewitt, University of Washington. (Address provided under the Visiting Lectureship Program of the Mathematical Association of America.)

H. M. Cox, *Secretary*

THE APRIL MEETING OF THE OHIO SECTION

The forty-first annual meeting of the Ohio Section of the Mathematical Association of America was held at The University of Cincinnati, Cincinnati, Ohio, on April 20, 1957. Professor P. V. Reichelderfer, Chairman of the Section, presided at the morning and afternoon sessions. There were 61 persons in attendance including 53 members of the Association.

The following officers were elected for the coming year: Chairman, Professor Samuel Selby, University of Akron; Secretary-Treasurer, Professor Foster Brooks, Kent State University; Third member of the Executive Committee, Professor Andrew Sterrett, Jr., Denison University; Program Committee: Chairman, Professor E. B. Leach, Case Institute of Technology, Professor Samuel Selby, University of Akron, and Professor H. E. Tinnappel, Bowling Green State University.

The Section voted that it is interested in sponsoring the Association high school mathematics contest in this region, and authorized the establishment of a standing committee to assume local responsibility.

The following papers were presented:

1. *On the geometrical meaning of a derivative*, by Professor P. V. Reichelderfer, The Ohio State University. (Chairman's address.)

Let T be a continuous transformation from a bounded domain in n-space into a bounded portion of n-space which is essentially of bounded variation. If C is a maximal model continuum for the point TC under T then for every sufficiently small open n-cube Q containing TC there is a unique component D of its inverse under T containing C, for which the topological index $u(x, T, D)$ is defined almost everywhere and integrable. The limit of the ratio of the integral of $u(x, T, D)$ to the measure of D as Q closes down on TC is described.

2. *Eigenvalue computation by successive suppression*, by Professor R. F. Rinehart, Case Institute of Technology.

Known methods of iteration provide in rather simple fashion the dominant eigenvalue(s) and corresponding eigenvector(s) of a square matrix over the complex field. Methods of calculating the remaining eigenvalues and vectors are less elementary. By appropriate employment of the Frobenius theorem on the eigenvalues and vectors belonging to a polynomial in a matrix, it is possible to replace successively the dominant eigenvalue(s) by zero(s), and, in effect, to put a new eigenvalue in the position of dominance. This permits the re-application of the iterative (or other) process to obtain ultimately all eigenvalues and vectors.

3. *A serial numbering system for permutations*, by Mr. A. J. Gruber, Student, Kent State University.

The paper presents a simple system for establishing a bi-unique correspondence between the set of $n!$ permutations of n distinct objects taken n at a time, and the set of natural numbers from 1 to $n!$ Given one of the permutations of n objects, it is possible to calculate the number from 1 to $n!$ associated with the permutation; and, conversely, given n and an integer from 1 to $n!$, it is possible to calculate the corresponding permutation. Some of the algebraic properties of this system and some of the problems solved with the system were also discussed.

4. *Calculation of the probability of a bivariate normal distribution over a circular region*, by Mr. H. E. Fettis and Dr. H. L. Harter, Aeronautical Research Laboratory, Wright Air Development Center, Wright-Patterson Air Force Base, presented by Mr. Fettis.

Various analytical expressions are derived for obtaining the probability of a distribution with a density given by

$$z(x, y) = \frac{1}{2\pi\sigma_x\sigma_y} \exp\left\{-\frac{1}{2}\left[\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2}\right]\right\},$$

where the region is the one bounded by the xy-plane, the surface defined by the above expression, and a cylinder of radius K_{σ}. These are well suited to either a desk or automatic digital computer.

Numerical results are presented giving those values of K for which $P(K, \sigma_x, \sigma_y) = \int z(x, y)dxdy = .5$, where the integral is taken over the region $(x^2+y^2)^{1/2} \leq K\sigma_x$. These calculations were carried out on the Burroughs E-101, a small-scale digital computer.

5. *Synthesis of electric circuits as a problem in mathematical logic*, by Professor Alonzo Church, Princeton University.

Propositional calculus is used in treating circuits in which action time of the elements may be neglected. Analogue of this for sequential circuits is *restricted recursive arithmetic*, a system related to Skolem's *recursive arithmetic*. A given condition which a circuit is to satisfy may be expressed in

logistic notation, with aid of quantifiers *etc.* if necessary. The *synthesis problem* is then to find an algorithm for obtaining recursion equivalences defining the required circuit. The general solution has been found in important special cases and solution of others seems hopeful. Adaptation of the method to asynchronous circuits is possible but cumbrous.

6. *A new approach to the teaching of elementary mathematics*, by Professor B. H. Gundlach, Bowling Green State University.

A college course in mathematics for elementary teachers and the program for in-service teachers which forms an intrinsic part of it are described. Basic concepts used are those of transformation and invariance. The desirable trait to be developed in students is transform ability. The psychological approaches of association and Gestalt theories are combined to form the dynamics of the advocated approach. Emphasis is placed upon (1) a mathematically *continuous* program, (2) the *creative* aspects of elementary mathematics, and (3) *construction* of problems rather than solving. The classrooms of the in-service teachers form the life laboratories for the college trainees.

7. *Mathematics in the Cincinnati Public High Schools*, by Miss Mildred Keiffer, Cincinnati Board of Education.

All pupils in Cincinnati Public Schools are required to study mathematics in grades 7 and 8 and a minimum of one year in mathematics in grades 9 through 12. The latter must be algebra unless the pupils score below the critical score on an eighth grade mathematics test. All college preparatory students are required to study a minimum of two years of mathematics, algebra and plane geometry. Advanced standing classes for selected pupils start in grade 10. Approximately 45% of all high school graduates study three or more years of mathematics, and about 60% of all college-bound students study mathematics for three or more years.

The acute shortage of teachers with an adequate background of mathematics is a real problem. For example, the Cincinnati Public Schools must fill 25 vacancies for next year, several of which are for advanced mathematics.

FOSTER BROOKS, *Secretary*

THE APRIL MEETING OF THE OKLAHOMA SECTION

The semi-annual meeting of the Oklahoma Section of the Mathematical Association of America was held at the University of Arkansas, Fayetteville, Arkansas, April 12–13, 1957. Professor O. P. Sanders, Chairman of the Section, presided. There were 42 persons in attendance including 27 members of the Association.

Copies of the employment register from recent meetings were available. Several informal luncheon discussions were arranged by congenial groups. The section experimented by permitting more time for individual papers at this meeting. The minimum time allotted was 30 minutes. Time was also arranged for informal discussion.

The ladies of the staff and distaff of the University of Arkansas served homemade sandwiches, cookies, rolls and coffee during the refreshment breaks each day.

The following discussion groups and papers were presented:

Friday, April 12:

1. Informal luncheon and discussion.

First session for contributed papers, Professor O. P. Sanders, Chairman, presiding.

2. *Functions with positive real part*, by Professor Harold Shniad, University of Arkansas.

A polynomial $f(z)$ of degree n of the form $f(z) = (1-z)\sum_{k=0}^{n-1} a_k z^k$ is shown to have positive real part in the unit circle $|z| \leq 1$ if and only if a related polynomial of degree $(n-1)$ has positive real part in the unit circle. The related polynomial $g(z)$ is given by

$$g(z) = \sum_{k=0}^{n-1} a_k + 2\sum_{k=0}^{n-1} a_j \cdot z^k.$$

Applicability of this result to extremal problems in coefficients of univalent functions is discussed.

3. *Topological vector lattices*, by Mr. J. C. Bradford, University of Oklahoma.

A locally convex topology on a vector lattice is said to be *compatible* if it has a basis consisting of ordered closed sets, *i.e.* sets for which $x \in V$, $|y| \leqq x \Rightarrow y \in V$. It is found that every total order-closed subspace of the order dual is a topological dual for some compatible topology. The finest compatible topology is the *bornological* topology whose basic system of bounded sets is the class of all order-bounded sets. Hence the finest compatible topology is the *Mackey* topology for the order dual. If there is a *complete* bornological compatible topology, it is unique and is the finest compatible topology. (This generalizes a theorem of C. Goffman, *Proc. Nat. Acad. Sc.* U.S.A., vol. 42, 1956.)

4. *A new kind of truth table*, by Professor W. E. Stuermann, University of Tulsa.

This truth table consists of an array of line segments (a line segment is assigned to each variable and its negation) arranged so that a "compartment" is defined for each of the possible sets of values which can be assigned to the variables. The use of the table was demonstrated for functions in disjunctive normal form, in conjunctive normal form, and in a form where some clauses are conjuncts of variables (or their negations). The technique is simpler and quicker, especially for functions of more than two variables, than is the use of the conventional truth table. The paper also mentions the use of this procedure as a graphical method for simplifying truth functions.

5. *Business Meeting.*

6. *Three hyperbolas associated with a triangle*, by Professor Emeritus N. A. Court, University of Oklahoma. (This MONTHLY, vol. 64, 1957, pp. 241–247.)

7. *Current availability of institutes and other supported programs for training teachers of mathematics*, by Professor J. H. Zant, Oklahoma State University of Agriculture and Applied Science.

Government money through the National Science Foundation to the amount of nine million dollars is being used to support high school and college teachers of Mathematics and Science in 95 Summer Institutes and 16 Academic Year Institutes for the period June 1957 to June 1958. In addition numerous grants from industry and private foundations are being used for similar purposes. Mathematics content, as well as science content, for these programs vary widely from review courses in traditional subject matter to thoroughly modern courses and sets of lectures in the various fields. This financial support and the opportunity for the study of content on an advanced level furnish a challenging and stimulating experience for teachers in service who wish to improve their competence in the sciences. Too little is being done to show teachers just how modern concepts in Mathematics and Science can be used in actual classroom situations. Steps are already being taken to remedy this situation. However, with the rapid expansion of these programs there is a real danger that not enough competent scientists who have ability in subject matter and interest in teaching problems can be found to man the programs.

8. Informal discussion and time for examination of the Employment Register.

An Arkansas barbecued chicken dinner was served at the Faculty Club, following which discussions and demonstrations of the mechanics of rolling spheres and distribution functions for normal cards were presented, along with an open discussion of current trends in college mathematics.

Saturday, April 13:

Second session for contributed papers, Professor O. P. Sanders, Chairman, presiding.

9. *Application of the continuously-compounded interest law to the dissipation of moisture*, by Mr. K. C. Cartwright, Vandervoort, Arkansas, introduced by the Secretary.

It is believed that when initial conditions sufficient for the solution of differential equations are known but the solutions of the equations too difficult, then it is possible, in some cases, to rely

on the continuously-compounded interest law to obtain a result sufficiently accurate for technical purposes. One such problem is the dissipation of moisture over a farm field subsequent to irrigation. Only end conditions are necessary and the need for physical constants needed for differential equations solution is eliminated. It is pointed out that cases of application are frequent enough to keep the continuously-compounded interest law always in mind for possible application when necessary.

10. *Dynamical trajectories and union curves*, by Professor C. E. Springer, University of Oklahoma.

In a paper, *Dynamical trajectories and geodesics*, Ann. of Math., vol. 30, 1929, pp. 591–606, Eisenhart showed that the trajectories of a mechanical system of n degrees of freedom correspond to the geodesics of a Riemannian space of $n+2$ dimensions in case the potential function involves the time, and that the Riemannian space has $n+1$ dimensions in case the potential function does not involve the time. Professor Springer developed the equations of union curves and showed that for the case in which the potential function does not involve the time, the dimensionality $n+1$ may be reduced to n if the geodesics are replaced by union curves with respect to a certain congruence.

11. *Some properties of non-continuous transformations*, by Professor O. H. Hamilton, Oklahoma State University of Agriculture and Applied Science.

Special types of noncontinuous transformations are defined and discussed. Among these are connectedness maps and peripherally continuous transformations. It is shown that under certain conditions a connectedness map is peripherally continuous, and that under certain other conditions a peripherally continuous transformation of a compact continuum leaves a point of that continuum fixed. Some ways in which noncontinuous transformations may arise from continuous transformations are discussed. Several examples of noncontinuous functions and transformations are given.

12. *Existence of surfaces in a certain conformal correspondence*, by Professor T. K. Pan, University of Oklahoma. (Presented by title).

Let two surfaces S and $\bar{S}$, associated with unit vector fields v and $\bar{v}$ respectively, be in conformal correspondence such that: (1) the curve of v corresponds to the curve of $\bar{v}$, (2) the asymptotic line of v corresponds to the asymptotic line of $\bar{v}$, and (3) the geodesic curvature of v along any curve C on S is equal to the geodesic curvature of $\bar{v}$ along the corresponding curve $\bar{C}$ on $\bar{S}$. This paper investigates the existence of these surfaces. Exterior differential calculus and the method of moving trihedrals are employed. It is found that the general solution depends on six arbitrary functions of one variable. Singular solutions and Cauchy problem are discussed.

13. *Boolean algebra and simple switching circuits*, by Professor W. E. Stuermann and Professor Simon Green, University of Tulsa.

This paper begins by developing Boolean algebra as an abstract system. The postulates are listed and certain derivable theorems are presented. The system is then restricted to a two-valued algebra. The peculiar characteristics of this kind of algebra are cited. The system is then interpreted in terms of switching circuits. Methods of simplifying functions are applied to the problem of simplifying circuits. The relations between switching operations, circuits and Boolean functions are described.

14. *On mappings on spheres*, by Professor J. W. Keesee, University of Arkansas, introduced by the Secretary.

A combinatorial lemma of Tucker and Fan is used to prove several theorems about sphere mappings.

15. Informal luncheon and discussion.

R. V. ANDREE, *Secretary*

THE APRIL MEETING OF THE SOUTHWESTERN SECTION

The annual meeting of the Southwestern Section of the Mathematical Association of America was held at the University of Arizona, Tucson, Arizona, on April 26–27, 1957. Professor R. B. Lyon, Chairman of the Section, presided at the afternoon session on April 26, and also at the morning session on April 27. There were 60 persons in attendance including 39 members of the Association.

The following officers were elected: Chairman, Dr. R. C. Hildner, Sandia Corporation, Albuquerque; Vice-Chairman, Professor J. H. Butchart, Arizona State College, Flagstaff; Secretary-Treasurer, Professor Deonisie Trifan, University of Arizona.

The following papers were presented:

1. *The equivalence of the method of Stodola to the use of Green's function in solving Sturm-Liouville systems*, by Professor L. C. Barrett, Arizona State College, Tempe, and Professor C. R. Wylie, Jr., University of Utah, presented by Professor Barrett.

The practical solution of eigenvalue problems by iterative procedures based on the so-called method of Stodola and on the use of "influence-functions," *i.e.* Green's functions, is commonplace in applied mathematics. However, it does not seem to be well known that these two methods are in fact completely equivalent, at least in the case of quite general second and fourth order Sturm-Liouville differential systems. It is the purpose of the present paper to point out this equivalence.

2. *The cardinal number of a metric space defined on* (0, 1), by Professor H. D. Sprinkle, University of Arizona, introduced by the Secretary.

Assuming that $2^{\aleph_0}=c$, it is proved that a certain metric space—based upon equivalence classes of subsets of (0, 1)—has the cardinal number $2^c=f$.

3. *Sides, angles, and faces of progressions*, by Professor J. H. Butchart, Arizona State College, Flagstaff.

Professor Butchart extended the property that if the sides of a triangle form an arithmetic progression then the join of the centroid and the incenter is parallel to a side. Using position vectors, he showed that the join of the centroid and the orthocenter (circumcenter, symmedian point) is parallel to a side if the tangents of the angles (sines of twice the angles, the squares of the sides) form an arithmetic progression. Also, the join of the centroid and the incenter of a tetrahedron is parallel to a face if the faces form an arithmetic progression.

4. *A characterization of conics*, by Mr. Louis Child, New Mexico College of Agriculture and Mechanic Arts.

At a point V of maximum curvature on a conic γ of eccentricity e, all j-th derivatives ρ_j of the radius of curvature ρ with respect to the arc-lengths vanish for j odd; for $j=2n$, $\rho_j=[3e^2P_j(e^2)]\rho^{1-j}$ where $P_j(e^2)$ is a polynomial in e^2 of degree $(j-2)/2$ with coefficients integers. For $j=2$, $\rho\rho_2=3e^2$.

5. *Periodic sets for second order differential equations with periodic forcing functions*, by Professor E. D. Nering, University of Arizona.

Let $D(x)=\sin t$ be a second order differential equation for x as a function of t with the periodic forcing function $\sin t$. For any given solution $x(t)$ of this equation, the set of points $[x(t_n), \dot{x}(t_n)]$ where $t_n=2n\pi$, n running through the set of integers, is called a periodic set. Using an analogue computer these points can be easily plotted. Periodic sets for some elementary nonlinear second order differential equations have been plotted, in particular for Daffing's equation. The resulting patterns are strikingly systematic and have not yet received a satisfactory explanation. Some conjectures which they suggest can be easily verified for the linear case.

6. *Some remarks on monomial groups*, by Mr. A. B. Gray, Jr., New Mexico College of Agriculture and Mechanic Arts, introduced by the Secretary.

Let $\sum_n(H)$ be the monomial group of H over U where H is a group and U is a finite set.

The normal subgroups of $\sum_m(H)$ are known (see: O. Ore, *Theory of Monomial Groups*, Trans. Amer. Math. Soc., vol. 51, 1942, pp. 15–64). The set of all monomial substitutions with permutation component in the alternating group is a subgroup $\sum_{A,m}(H)$ and the normal subgroups have been determined for $m \geqq 5$, $m = 2$. (See: R. B. Crouch, *Monomial Groups*, Trans. Amer. Math. Soc., vol. 80, pp. 187–215.) The problem of determining the normal subgroups when $m = 3, 4$ is discussed.

7. *Preliminary report on the definition of the system concept*, by Dr. D. O. Ellis, Litton Industries, Beverly Hills, California.

Although numerous theories of special systems and a few of relatively general systems have been proposed, there is, in the writer's knowledge, no theory encompassing those things generally referred to as systems and/or operations and meeting the basic criteria: (1). potential of specialization to any desired degree, (2). avoidance of restricting notions inherently "non-realizable physically" (*e.g.*, the infinite tape concept of Turing's theory), and (3). sufficient generality to permit coverage of all presently known and predicated types of information processing.

Presented in this communication is a tentative definition of the concept upon which such a theory would be based. It is believed the definition permits satisfaction of the three criteria above and yet is not so general as to be meaningless. The notions employed are elementary ones from set theory and lattice theory.

8. *Optimum rocket motion and the satellite problem*, by Professor L. C. Barrett, Arizona State College, Tempe, and Dr. Herbert Knothe, Holloman Air Development Center, Holloman, New Mexico, presented by Professor Barrett.

This paper is concerned with optimum problems of the following type:

"Let a rocket, with an arbitrary initial velocity, be required to expend its fuel continuously during the course of a given flight time. Moreover, suppose that at the instant the flight time terminates the rocket is to have a prescribed final velocity. Of all paths the rocket may pursue in fulfilling these conditions we desire the one(s) over which a minimum of fuel is required."

A path thus characterized is defined as an optimum trajectory corresponding to the prescribed velocity conditions. In addition to giving a necessary and sufficient condition for optimum rocket motion in a constant gravity field, we also find necessary conditions for optimum rocket motion in a general central force field.

9. *The mathematics curriculum and the computer*, by Dr. H. R. J. Grosch, General Electric Company, Tempe, Arizona.

10. *The geometry of variation of parameters*, by Professor R. M. Conkling, New Mexico College of Agriculture and Mechanic Arts.

The effect of perturbation of a homogeneous second order differential equation is studied through the change in shape of its solution in the phase plane. The variation of parameters technique for finding a particular integral of the perturbed equation can be interpreted as a change of coordinates in this phase plane, and the usual conditions imposed on the parameters are easily explained in this setting.

11. *Ways and means for the solution of boundary value problems in engineering*, by Dr. M. A. Dengler, AiResearch, Phoenix, Arizona.

Most of the nonlinear partial differential equations in engineering can be linearized by maintaining the coefficients of the second derivatives of the unknown function as functions of the independent variables of the system, and following a scheme of consecutive iterations. The functions indicated are kept constant during one phase of the iteration and changed from phase to phase. In this perspective also nonlinear equations of the second order can be subdivided into the categories of elliptic, parabolic and hyperbolic equations. Methods for solving all three types by finite

differences and matrix iteration were illustrated. Methods are specifically suited to modern means of high speed computation.

12. *Model making and the computer*, by Dr. H. R. J. Grosch and Dr. F. B. Thompson General Electric Company, Tucson, Arizona.

13. *Subordination in complex variable theory and a theorem on minimal surfaces*, by Mr. J. H. Evans, Los Alamos Scientific Laboratory, introduced by the Secretary.

A definition of subordinate functions is given, and theorems concerning them by G. M. Goluzin and E. Reich are stated. A definition of minimal surfaces is stated and a convenient notation for their representation is presented. Then the following theorem is proved for pairs of minimal surfaces whose coordinate functions are subordinate: *If f_j and F_j are the coordinate functions of two minimal surfaces given in conformal representation and f_j is subordinate to F_j, then the minimal surfaces are plane surfaces and the F_j surface is a subset of the f_j surface.*

14. *On two lemmas of F. Riesz*, by Professor Oswald Wyler, University of New Mexico.

The theory of Lebesgue integration can be based on two simple lemmas on integrals of step functions, as shown in: F. Riesz and B. Sz.-Nagy, *Functional Analysis*, New York 1955 (*cf.* pp. 30 and ff.). A proof of these two lemmas is given for the most general case that a measure μ is defined as a countably additive set function in a ring $\mathfrak{R}$ of sets, a step function being defined as a linear combination of characteristic functions of sets in $\mathfrak{R}$.

15. *Guessing on multiple choice tests*, by Professor R. F. Graesser, University of Arizona.

If a student guesses at random on a multiple choice test, the expected value of his score is zero, when his score is obtained by the usual formula. It can be proven that the odds never favor a guesser on a true-false test. However, if there are 3, 4, or 5 answers from which to choose, then an investigation of 190 special cases yields the empirical rule that the odds favor the guesser if, and only if, the number of questions plus one is divisible by the number of choices.

16. *Principal ideal domains with no division algorithm*, by Professor D. W. Dubois and Dr. Arthur Steger, University of New Mexico, introduced by the Secretary.

It is already known that the ring of algebraic integers in $K(\sqrt{-m})$ (K is the field of rationals) with m a positive, square-free integer, is a principal ideal domain when $m=19$, 43, 67, and 163, but the domain is not euclidean according to the norm. The case $m=19$ was known to Dedekind. In this paper we show that none of the above domains is euclidean by *any* function. The methods do not apply to real quadratic fields.

W. W. MITCHELL, JR., *Secretary*

THE APRIL MEETING OF THE TEXAS SECTION

The spring meeting of the Texas Section of the Mathematical Association of America was held at the University of Houston, Houston, Texas, on April 26–27, 1957. Professor Don Cude and Professor Martin Wright presided at the regular sessions. There were 120 persons in attendance, including 80 members of the Association.

At the business meeting the following officers were elected for the year 1957–58: Chairman, Professor Martin Wright, University of Houston; Vice-Chairman, Professor E. K. McLachlan, Baylor University; Secretary-Treasurer, Professor C. R. Sherer, Texas Christian University.

The following papers were presented:

1. *Some loci of the Lemoine point of a triangle*, by Professor C. P. Benner, University of Houston.

Let P be any point in the plane and let L be the Lemoine point of a certain triangle associated

with P. If P is allowed to vary according to specified restrictions, the Lemoine point describes, among other loci, the four-leaf rose, the lemniscate, the cissoid of Diocles, and the bifolium.

2. *Wronskian determinants and linear dependents*, by Professor W. T. Guy, Jr., University of Texas.

Students seem to have a rather difficult time learning the correct relationships between Wronskian determinants and linear dependence. An alternative determinant is suggested which might be considered as a more natural criterion for linear dependence.

3. *8-Rings in minimal maps*, by Mr. B. T. Goldbeck, Jr., Texas Christian University.

For an irreducible ring of eight regions, three sets of criteria are established. These criteria are Kempe equalities (E), primary inequalities (P), and secondary inequalities (S). Each peripheral 8-ring, not known to be reducible, was tested by these criteria. The entire class of peripheral 8-rings were shown to be irreducible, (E), (P) and (S). Algebraic cases were constructed, which though undrawable, also satisfied these 1,180 criteria.

4. *Prime divisors of sequences of integers*, by Professor L. K. Durst, Rice Institute.

The Fibonacci numbers are given by the recursion $F_n=F_{n-1}+F_{n-2}$, $F_0=0$, $F_1=1$. For each n except 6 and 12 there is a prime p such that $p\mid F_n$ but $p\nmid F_m$, $0<m<n$. The Fibonacci sequence F_n is a special Lehmer sequence P_n defined by the recursions $P_{2n}=P_{2n-1}-MP_{2n-2}$, $P_{2n+1}=LP_{2n}-MP_{2n-1}$, $P_0=0$, $P_1=1$, $L>0$. Suppose $(L, M)=1$ and $K=L-4M>0$. An index n greater than 2 is called *exceptional* if $p\mid P_n$ implies $p\mid\prod_{m=1}^{n-1}P_m$. For these Lehmer sequences the following results are proved. The only exceptional indices are 6 and 12. The index 12 is exceptional only when $L=1$, $M=-1$ or $L=5$, $M=1$. The index 6 is exceptional if and only if K is odd, $L=2^{N+2}-3K>0$, $M=2^N-K$. Compare M. Ward, *Ann. of Math.* (2), vol. 62, 1955, pp. 230–236.

5. *Some formulas and congruences involving Bernoulli numbers and Fibonacci numbers*, by Mr. R. P. Kelisky, University of Texas.

The polynomials $F_n(x)$ which have the generating function $2e^{xt/2}\sinh(t/2)$ reduce to combinations of Fibonacci and Lucas numbers for certain values of x. Simple formulas expressing sums of products of binomial coefficients, Fibonacci numbers, and Bernoulli numbers in terms of Fibonacci and Lucas numbers may be obtained by multiplying the generating function for the sequence $F_n(x)$ by the well-known generating function for the Bernoulli polynomials. From these formulas are obtained several congruences with respect to a prime modulus involving the numbers of Bernoulli, Fibonacci, and Lucas.

6. *Analytic functions of a bicomplex variable*, by Professor G. B. Price, University of Kansas. (By invitation.)

Let a and b be complex numbers. It has been customary to call $a+jb$ a bicomplex number if addition, subtraction, multiplication, and division are formally the same as for the complex numbers, with $j^2=-1$. The bicomplex numbers contain divisors of zero. This paper contains an outline of a theory of analytic functions of a bicomplex variable $x+jy$. The theory has striking similarities and dissimilarities with the theory of functions of a complex variable.

7. *A class of k-sets of polynomials*, by Professor F. J. Palas, Southern Methodist University.

A k-set of polynomials is defined as a sequence of polynomials such that $P_{kn}(x)$ is of degree exactly kn. A linear differential operator is developed which has the property that $L[P_{kn}(x)]=P_{k,n-1}(x)$. A class of k-sets is characterized by the generating function, $g(x, t)=f(t)\exp[p(x)u(t)]$, where $p(x)$ is a polynomial of degree k such that $p(0)=0$, and $f(t)$ and $u(t)$ are power series. Several recurrence relations are developed and are used to classify the k-sets as B_k-type and C_k-type in analogy with Sheffer's B-type and C-type classes. The equivalence of these classes is shown.

8. *A generalization of Bernoulli numbers and Bernoulli polynomials*, by Professor L. L. Silverman, University of Houston.

To each system consisting of a function $f(x)$, analytic about the origin, and of a positive integer p, there is associated a function $B(x)$ which generates the generalized Bernoulli numbers. The Euler numbers, as well as the ordinary Bernoulli numbers, and other well-known sets of numbers are included as special cases. Corresponding to each system, $f(x)$ and p, there is also defined a set of polynomials, which includes as special cases well-known sets of polynomials. An interesting feature is the novel algebra connecting the generating functions and their corresponding associated functions $f(x)$.

9. *A non-reflecting solution of Maxwell's equations*, by Professor J. T. Hurt, Agricultural and Mechanical College of Texas, introduced by the Secretary.

The early examples of nonreflecting waves in nonhomogeneous media required the assumption of properties of the media which are physically impossible. A new example is given in a medium with properties which are physically possible.

10. *Axioms of a generalized projective geometry*, by Professor D. E. Edmondson, Southern Methodist University.

A generalization is given for the axioms of a projective geometry. The points constitute a partially ordered set. When there are no orderings among the points, the axioms become the usual Veblen-Young axioms of a projective geometry. From the axioms it follows that the lattice of subspaces is a modular lattice and that generalized projective geometries serve as representation spaces for modular lattices.

11. *Properties of the family of curves generated from a base curve $\rho_0=\rho_0(\theta)$ by the transformation $\rho_n=g^n(\theta)\rho_0$*, by Professor F. S. Nowlan, University of Illinois, and Mr. A. A. Aucoin, University of Houston, presented by Mr. Aucoin.

The family of curves $\rho_n=g^n(\theta)\rho_0$ generated from a base curve $\rho_0=\rho_0(\theta)$ is shown to have many algebraic invariants. Some of these invariants have simple geometric interpretation.

12. *On the Gaussian curvature of modular surfaces*, by Mr. Daniel Weiser, Rice Institute.

Consider an analytic function $w=f(z)$ which is, with its first two derivatives, not zero at z_0. If $K(z_0)$ is the Gaussian curvature at the point $P(z_0, |f(z_0)|)$ on the associated modular surface, then $K(z_0)$ is greater than, equal to, or less than 0 if and only if $\Re\{[f'(z)]^2/[f(z_0)f''(z_0)]\}$ is greater than, equal to, or less than 1. The principal result, however, is the evaluation of the Gaussian curvature at the zeros of $f(z)$ and its derivatives.

13. *Some probability distributions for sociometric matrices*, by Professor P. D. Minton, Southern Methodist University.

A "sociometric matrix" has elements representing data collected in certain sociological investigations. Probability distributions obtained under acceptable assumptions are required to analyse the data. Assuming elements 0 or 1 with the main diagonal null, and assuming random entries in each row with row totals fixed, probability distributions related to column totals are given. Equal and unequal row totals are considered. The joint distribution of two column totals and of the sum of two column totals and the conditional distribution of the difference of two column totals given their sum are obtained or approximations indicated.

14. *The plane in life and mathematics*, by Professor R. S. Underwood, Texas Technological College.

Although a large part of the heritage of mankind is stored on the flat pages of books, the branch of mathematics which uses a plane for illustrative purposes—namely, plane analytic geom-

etry—is decidedly skimpy in its coverage, being restricted to two variables. It is therefore contended that a plane analytic geometry for n variables deserves far more attention and priority as a goal than mathematicians as a group have thus far accorded it. The writer's lone trek on the road to this goal is reviewed in its broader aspects, and some specific new results are described.

15. *Mathematics in the general education program of colleges and universities in the United States*, by Professor W. I. Layton, Stephen F. Austin State College.

This study covers 150 colleges and universities in 42 states. It is an investigation of the mathematics required and recommended for the general education program. General mathematics is the course most frequently required and most highly recommended for this program. The mean of mathematics recommended is 4.82 semester hours. This is in contrast to a mean of 1.97 semester hours of required mathematics in colleges which have general education programs. The study also presents topics which might be included in college general mathematics.

C. R. SHERER, *Secretary*

THE MAY MEETING OF THE ILLINOIS SECTION

The thirty-sixth annual meeting of the Illinois Section of the Mathematical Association of America was held at Illinois State Normal University, Normal, Illinois on May 10 and 11, 1957. Professor F. E. Hohn, Chairman of the Section, presided at all sessions. There were 63 persons in attendance, including 55 members of the Association.

At the business meeting on Friday afternoon the following officers were elected to serve for the coming year: Chairman, Professor C. T. McCormick, Illinois State Normal University; Vice-Chairman, Professor A. E. Hallerberg, Illinois College; Secretary-Treasurer, Professor A. W. McGaughey, Bradley University.

Professor Joseph Stipanowich reported on the work of the Committee on Contests and Awards, stating that 4800 students from 120 different high schools participated this year, this being 1000 more than a year ago and 2000 more than two years ago. The Section then voted to cooperate with the National Committee on Contests in Mathematics for High School Students to the extent permitted by the Illinois High School Association. Professor Stipanowich pointed out that this Association will not approve the publication of scores made by the students until the contest has been approved by the National Association of Secondary School Principals, but his committee is working to get the approval of this Association.

Professor A. E. Hallerberg reported for the Committee on the Strengthening of the Teaching of Mathematics. Following his request for guidance of the Committee's efforts the Section authorized the Committee to pursue the idea of initiating a summer work camp in mathematics for high school students in Illinois, part of the necessary funds to come from the Section's treasury.

The following papers were presented:

1. *One hundred years of mathematics in Illinois*, by Professor Emeritus E. H. Taylor, Eastern Illinois State College.

The paper dealt with the preparation of elementary teachers and the teaching of arithmetic, influences leading to the improvement of the teaching of mathematics, and the increasing importance of mathematics and its teaching.

2. *Some comments on the finite induction postulate*, by Professor L. D. Rodabaugh, Southern Illinois University.

The paper presented a point of view on the Peano Postulates, particularly the Finite Induction Postulate, which may prove helpful to persons teaching courses in part dependent upon these postulates. In particular it is believed that the examples here provided will be pedagogically beneficial.

3. *Topics from matrix algebra suitable for college freshmen*, by Professor Arnold Wendt, Western Illinois State College.

Because most college freshmen have never encountered any but the real and complex number systems, they think that study of the postulates for these systems is a waste of time since these properties are obvious. The speaker has found 2×2 matrices can serve as an example of a non-commutative system with zero divisors which is simple enough to understand yet impressive enough to increase appreciation of the role of postulates and definitions.

Matrices can also be introduced quite naturally during the study of determinants and simultaneous linear equations.

4. *Experiences of a wandering mathematician*, by Professor W. W. Sawyer, University of Illinois, introduced by the Chairman. (By invitation.)

This paper presented some recollections of the English school system, allowing for very rapid progress of mathematically inclined students, its weakness with non-academic students; the New Zealand system as a contrast, satisfactory for least academic students, very unsatisfactory for scholars; lack of flexibility in both systems; no reason seen why an educational system should not cater to all types; some experiences in teaching English pupils of low intelligence, and the use of apparatus to secure interest.

An account was given of a high school mathematics society in New Zealand and its effect in increasing the supply of mathematics teachers.

5. *A supplementary training program for mathematics teachers*, by Professor Joseph Landin, University of Illinois.

The speaker outlined the supplementary training program for secondary school mathematics teachers at the University of Illinois. The basic courses of this program deal with the construction of the real number system beginning with the elements of set theory; elements of logic; axiomatic systems; analysis of high school geometry; Hilbert's axioms. Additional courses of the program are: the foundations of the calculus; modern algebra and geometry. Pedagogical methods and new curriculum developments in secondary school mathematics are studied, attention being given to the work of the University of Illinois Committee on School Mathematics.

6. *Problems encountered in industry by a junior mathematician*, by Professor W. J. Thomson, Western Illinois State College.

Three problems which the speaker encountered while working in an industrial position were presented. These problems were given to junior mathematicians to solve. The problems included one in which experimental data was to be tabulated and the results presented in usable form. Another involved the solution of a differential equation by numerical integration, and the third was concerned with determining an estimate of the maximum possible rates and accelerations from functions which did not lend themselves to direct solutions.

7. *Sequential machines*, by Professor D. D. Aufenkamp, University of Illinois.

Roughly speaking, sequential machines are devices designed to make certain sequences of "outputs" correspond to given sequences of "inputs." A simple, abstract model of many sequential machines has been proposed by G. H. Mealy. This model assumes finitely many "inputs," "outputs," and "states." Its present output and next state are determined uniquely by the present input and the present state. A representation of the model as a directed graph provides a convenient starting point for investigations into problems of analysis and synthesis of such machines. The speaker discussed several of these problems, and indicated how they could be solved.

8. *Geometry in transition*, by Professor D. R. Bey, Illinois State Normal University.

Evidence that the content and objectives of geometry are undergoing a period of adjustment is readily apparent. For example, the number of theorems considered basic to plane geometry

has been reduced from 212 to 30 in the past 25 years. Results of studies lend support to the soundness of the reasons for the present swing away from the "critical thinking in life situations" approach to geometry. Other findings suggest adding solid geometry, analytic geometry and topics from "other geometries" at the 10th grade level. The studies also recommend that geometry offerings at the college level should give first priority to a course in projective geometry and should include courses in noneuclidean geometries and fundamental concepts of geometry.

A. W. McGAUGHEY, *Secretary*

THE MAY MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The spring meeting of the Maryland-District of Columbia-Virginia Section of the Mathematical Association of America was held at the Johns Hopkins University, Baltimore, Maryland, on May 4, 1957. The meeting was sponsored jointly by the Association and the Baltimore, Maryland and Washington, D. C. Sections of the Society for Industrial and Applied Mathematics. Professor R. C. Yates, Chairman of the Section, Dr. J. K. Sterrett, Chairman of the Washington, D. C. Section of SIAM, and Dr. Michael Aissen, Chairman of the Baltimore, Maryland Section of SIAM, presided at joint sessions. There were 104 persons in attendance, including 84 members of the Association.

The following officers were elected to serve for a period of one year: Chairman, Professor R. P. Bailey, U. S. Naval Academy; Vice-Chairmen, Professor M. G. Humphreys, Randolph-Macon Woman's College and Professor C. A. Spicer, Western Maryland College; Secretary, Professor D. B. Lloyd, District of Columbia Teachers College; Treasurer, Professor T. W. Moore, U. S. Naval Academy.

The following contributed papers were presented:

1. *The Bureau of Ships computer program*, by Mr. A. E. Smith, Navy Department, Washington, D. C.

The Bureau of Ships is presently operating two Univacs at the David Taylor Model Basin and two IBM-650's at naval shipyards. A wide variety of problems in both engineering and management fields have been solved. These were concerned with nuclear reactor design, magnetic minesweeping, vibration of plates, determination of ships lines, pipe stress analysis and allocation of electronic equipment to ships. The last problem was formulated as a linear programming problem.

2. *The pseudoinverse of a rectangular or singular matrix and its application to the solution of systems of linear equations*, by Dr. T. N. E. Greville, Social Security Administration, Washington, D. C.

It is not generally known that E. H. Moore generalized the notion of inverse of a matrix to include all nonzero matrices, rectangular as well as square. The general solution of any system of linear equations which has an exact solution can be conveniently expressed in terms of this pseudoinverse. The additional fact that his procedure gives a best solution in the sense of least squares when there is no exact solution was noted for a restricted class of systems of Bjerhammar, who was unaware of Moore's work and did not appreciate the generality of his result.

3. *An interesting fourth order differential system*, by Mr. C. H. Murphy, Jr., Aberdeen Proving Ground, Maryland.

Although the theory of linear differential equations with constant coefficients is well known, most of the examples and problems given in texts have characteristic equations which are either first or second order or are easily factorable. In this paper an important subset of the set of all fourth order differential systems is described. The solution of members of this subset can be obtained by the solution of at most two quadratic equations instead of the usual quartic equation.

4. *Solution of the integrodifferential equation of transfer by successive approximations*, by Dr. P. M. Anselone, Radiation Laboratory, Johns Hopkins University.

The equations

$$\mu \frac{\partial I(\tau, \mu)}{\partial \tau} = I(\tau, \mu) - 1/2 \int_{-1}^{+1} I(\tau, \mu')d\mu', \quad \int_{-1}^{+1} I(\tau, \mu')\mu' d\mu' = F \text{ (a constant)},$$

where $0 \leqq \tau < \infty$ and $-1 \leqq \mu \leqq 1$, plus certain auxiliary conditions, define a classical problem in transfer theory. The existence and uniqueness of the solution was established by E. Hopf, who obtained the Neumann series solution of a related integral equation. The Wick-Chandrasekhar technique for approximating I involves replacing the two integrals above by sums corresponding to the $2n$ point Gauss quadrature formula. The resulting problem is solved to yield an approximation, I_n, to I. The "double-Gauss" formula, in which the n point Gauss formula is applied separately to each of the intervals $-1 \leqq \mu \leqq 0$ and $0 \leqq \mu \leqq 1$, also yields an approximation. The principal result obtained is that the sequence $\{I_n : n \geqq 1\}$ corresponding to the double-Gauss formula converges to I uniformly on each compact subset of the domain of I.

5. *Matrix analysis for production scheduling and inventory control*, by Professor D. N. Chorafas, Catholic University of America.

With the use of high speed electronic data processing systems, mathematical techniques which seem to be very complicated or unduly involved became of importance and interest for the solution of engineering production problems.

Matrix analysis can be used to advantage for the solution of problems in Engineering Production. Commodity requirements for the initial, the intermediate and the final steps can be set in the form of a rectangular matrix. Then with a simple matrix multiplication engineering management is able to study the input-output requirements of any production system.

The speaker discussed the mechanics of the method from the conception of the model to the data processing through an electronic digital computer and evaluated the method with respect to its potentialities for future application in industry.

6. *Intermittent rotors*, by Mr. Michael Goldberg, Bureau of Ordnance, Navy Department, Washington, D. C.

The shape of the least area which, when placed at random on a square lattice of points, always includes at least one of the points was shown by J. J. Shaffer and D. B. Sawyer to be a square to which has been added the areas included by two parabolic arcs, one on each of two opposite edges of the square. The speaker showed that this shape is one of a family of convex curves which may be rotated through all orientations in the plane while passing through at least three of the four vertices of a square. Extensions to a series of similar problems were indicated.

In addition, the following hour lectures were presented by invitation of the joint program committee:

1. *Geometry in the mathematics curriculum*, by Professor W. L. Chow, The Johns Hopkins University (auspices of MAA).

2. *Quaternions and Clifford numbers*, by Professor Marcel Riesz, Institute of Fluid Dynamics, University of Maryland (auspices of SIAM).

R. P. BAILEY, *Secretary*

THE MAY MEETING OF THE ROCKY MOUNTAIN SECTION

The fortieth annual meeting of the Rocky Mountain Section of the Mathematical Association of America was held at the Colorado School of Mines, Golden, Colorado, on Friday afternoon and evening and Saturday forenoon, May 3 and 4, 1957. Professor R. R. Gutzman, Chairman of the Section, presided at all three sessions. There were 116 persons registered for the meeting, including 68 members of the Association.

Officers elected at the meeting for 1957–1958 were: Chairman, Professor D. O. Patterson, Colorado State College; Vice-Chairman, Professor N. C. Hunsaker, Utah State Agricultural College; Secretary-Treasurer, Professor F. M. Carpenter, Colorado School of Mines.

The following papers were presented:

1. *On generalized Legendre polynomials*, by Professor Arne Magnus, University of Colorado, introduced by the Secretary.

A recurrence formula is developed for the polynomials $P_k = P_k(\phi_1, \cdots, \phi_n)$ defined by $[1+\phi_1 t+\cdots+\phi_n t^n](m/n) = 1+P_1 t+\cdots+P_k t^k+\cdots$ and application made to the polynomial solutions of the partial differential equation $u_x \cdot v_y - u_y \cdot v_x = 1$ where $u = u(x, y)$ and $v = v(x, y)$.

2. *The Laplace transform in discontinuous solutions of a partial differential equation*, by Professor V. W. Bauman, Colorado School of Mines, introduced by the Secretary.

Using the Laplace transform to solve problems involving the equation (1) $Y_{tt}(x, t) = a^2 Y_{xx}(x, t)$, equation is assumed to be (2) $s^2 L\{y\} = a^2(\partial^2/\partial x^2)[L\{Y\}]$. In some problems the solution, $Y(x, t)$, or its partial derivatives of first order are only sectionally continuous. In this case the members of (2) are not the transforms of members of (1), both members of (2) having additional terms which involve the salti in the discontinuous functions. It was shown in this paper that the true transformed equation always reduces to equation (2) if the solution or its partial derivatives of first order have only finite discontinuities on the lines in the xt-plane, $t = C \pm x/a$.

3. *The economic index numbers of Divisia*, by Professor E. A. Davis, University of Utah.

An expository account of the "historical" index numbers, for prices and quantities of goods traded, due to Divisia (F. Divisia, *Economique Rationelle*, Paris, 1928) was presented. Relationships between these quantities and the index formulae of Laspeyres were noted and, in particular, a device for approximating the former by means of products of Laspeyres indexes was indicated.

4. *A mathematical analysis of fuel burnout in nuclear reactors*, by Captain A. W. Banister, United States Air Force Academy, introduced by the Secretary.

Mathematical analysis of nuclear reactors is necessary in order to provide quantitative information regarding certain design and operational features. One such item is the effect of fuel burnout in modern reactors operating at high power levels. An approach to this problem can be made by writing a differential equation descriptive of a generalized reactor volume element, and introducing perturbations in the reactor constants to simulate burnout. By making certain approximations justifiable on physical grounds, the equation can be transformed into a linear, non-homogeneous type, easily solvable by the operator method. The solution then provides quantitative information on changes in the power distribution function, and adjustments necessary to maintain level operation.

5. *A characterization of n-groups*, by Professor D. W. Robinson, Brigham Young University.

The generalized groups defined by W. Dörnte (*Untersuchungen über einen verallgemeinerten Gruppenbegriff*, Math. Z., vol. 29, 1928, pp. 1–19) are systems of elements with a polyadic operation satisfying an extension of the associativity and solvability axioms for ordinary groups. This note points out that these systems can be characterized as well by replacing the solvability axiom with a generalization of the identity-inverse axiom for groups.

6. *Infinite exponentials*, by Professor W. J. Thron, University of Colorado.

For every $n > 1$ let $t_n(z) = e^{a_n z}$, where the a_n are complex numbers. Define $T_n(z)$ to be: $T_1(z) = t_1(z)$, $T_n(z) = T_{n-1}(t_n(z))$. Then $\{T_n(1)\}$ is called an infinite exponential. It is proved that an infinite exponential converges if $|a_n| \leq e^{-1}$ for all n.

7. *Speed-up college mathematics*, by Professor I. L. Hebel, Colorado School of Mines.

A review of the progress of a highly selected group of twenty entering freshmen who have been allowed to progress at an accelerated pace with a view to completing 21 semester hours of college mathematics (through Differential Equations) in three semesters. From the results of this first attempt to do something for the better-than-average student, the conclusion reached is that many additional students taking the "standard" courses could profitably be placed in the accelerated program. It is anticipated that about 50 of the next group of 300 entering freshmen will be assigned to a similar group.

8. *Antenna theory*, by Dr. James Wait, Boulder Laboratories, National Bureau of Standards. (Invited Address.)

The calculation of the radiation field of a flush mounted antenna in the tangent plane (the classical light-shadow boundary) is not readily treated by either geometrical optics or the residue-series. In the former case the field is indeterminate and in the latter case the convergence is extremely poor and would actually diverge in the illuminated region. Despite the fact that the harmonic series is cumbersome, it is valid in this transition zone between the illuminated and shadow regions of space. Therefore, it is desirable to attempt to adapt the harmonic-series representation to surfaces of large radius of curvature. This is the purpose of the present paper.

9. *Orthogonal functions whose k-th derivatives are also orthogonal*, by Professor F. M. Stein, Colorado State University.

Several sets of orthogonal functions possess the property that the k-th derivatives of these functions form sets of orthogonal functions, perhaps with a different weight function, $w_k(x)$, but over the same interval. This paper shows that the set $\{1, \cos nx, \sin nx\}$ possesses this property. Also, if the orthogonal functions are polynomials they must be those of Hermite, Jacobi, or Laguerre; and these are the only polynomials possessing this property under the definition of orthogonality that for $\{\phi_n(x)\}$,

$$\int_a^b w(x)\phi_n(x)\phi_m(x)dx = c_n\delta_{mn}.$$

10. *Introduction to SOMAC*, by Professor R. R. Gutzman, Colorado School of Mines.

The speaker explained the basic operating exponents of the analog computer. He discussed the methods of programming a differential equation of the type

$$a_0\frac{d^2y}{dx^2} + a_1(dy/dx)^2 + a_2y = F(x),$$

and simultaneous linear algebraic equations. Different ways of generating $F(x)$ were considered.

11. *The relation of regular semigroups to groups*, by Mr. H. G. Moore, University of Utah.

In this paper theorems are given which relate regular semigroups and inverse semigroups with groups. A regular semigroup with cancellation is a group as is every regular semigroup generated by a single element. Cancellation may be relaxed slightly if certain other conditions are imposed on the regular semigroup. Every regular semigroup possesses subsets which are groups, and if not all the elements of the semigroup are idempotent it possesses nontrivial subsets which are groups.

12. *Likes and dislikes—when and why?*, by Professor O. M. Rasmussen, University of Denver.

A preliminary report of a survey in progress as an attempt to find when and why students like or dislike mathematics. Thus far it appears that grades 7, 9, and 10 are the critical ones with the teachers receiving credit or blame in most cases. Parental encouragement appears to be a factor

among those who now like mathematics. Information reported is from a survey of students at all levels in a university. Therefore, the results of changing conditions of the past five years are not included.

13. *The new Bachelor of Science Degree in Applied Mathematics at the University of Colorado*, by Professor L. W. Rutland and Professor J. R. Britton of the University, presented by Professor Rutland.

An announcement was made of the new degree being offered in the Department of Applied Mathematics at the University of Colorado. The curriculum includes study in the basic engineering sciences and applied mathematics subjects. The applied mathematics work includes intermediate differential equations, advanced calculus (*e.g.*, Taylor, *Advanced Calculus*), introduction to applied mathematics (*e.g.*, Wylie, *Advanced Engineering Mathematics*), statistics, algebraic methods, computing machines, and a senior seminar.

F. M. CARPENTER, *Secretary*

CALENDAR OF FUTURE MEETINGS

Forty-first Annual Meeting, University of Cincinnati and Hotel Sheraton-Gibson, Cincinnati, Ohio, January 31, 1958.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Washington and Jefferson College, Washington, Pennsylvania, May, 1958.

ILLINOIS, Illinois College, Jacksonville, May 9–10, 1958.

INDIANA, DePauw University, Greencastle, October 18, 1957.

IOWA

KANSAS

KENTUCKY, University of Kentucky, Lexington, April, 1958.

LOUISIANA-MISSISSIPPI, Loyola University, New Orleans, February 21–22, 1958.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Georgetown University, Washington, D. C., December 7, 1957.

METROPOLITAN NEW YORK

MICHIGAN, University of Michigan, Ann Arbor, March, 1958.

MINNESOTA, State Teachers College, Mankato, October 5, 1957.

MISSOURI, University of Missouri, Columbia, Spring, 1958.

NEBRASKA, University of Nebraska, Lincoln, April 19, 1958.

NEW JERSEY, Fairleigh Dickinson University, Rutherford, November 2, 1957.

NORTHEASTERN, Dartmouth College, Hanover, New Hampshire, November 30, 1957.

NORTHERN CALIFORNIA, San Francisco State College, January 18, 1958.

OHIO, Denison University, Granville, April, 1958.

OKLAHOMA, Oklahoma City University, October 25, 1957.

PACIFIC NORTHWEST

PHILADELPHIA, November 30, 1957.

ROCKY MOUNTAIN, Colorado State College, Greeley, Spring, 1958.

SOUTHEASTERN, University of Florida, Gainesville, March 14–15, 1958.

SOUTHERN CALIFORNIA, Pasadena City College, March 8, 1958.

SOUTHWESTERN, University of New Mexico, Albuquerque, April 11–12, 1958.

TEXAS, Baylor University, Waco, April, 1958.

UPPER NEW YORK STATE, Ecole Polytechnique and University of Montreal, Montreal, Quebec, Canada, May, 1958.

WISCONSIN, Carroll College, Waukesha, May, 1958.

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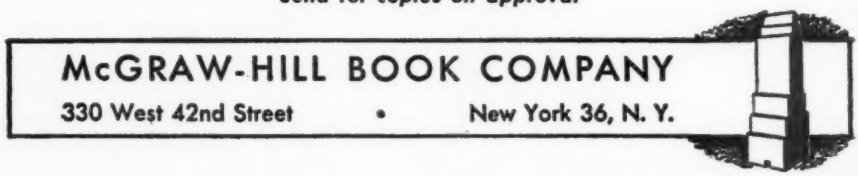

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THE
OTTO DUNKEL MEMORIAL
PROBLEM BOOK

Edited by

HOWARD EVES AND E. P. STARKE

This publication has been financed by the Otto Dunkel Fund
of the Mathematical Association of America

Published as a supplement to the AMERICAN MATHEMATICAL MONTHLY
Volume 64 Number 7
August–September, 1957

OTTO DUNKEL

FOREWORD BY THE EDITORS

When Professor Otto Dunkel died in 1951, it was found that he left a bequest to the Mathematical Association of America. After deliberation, the Board of Governors of the Association voted that the income from Professor Dunkel's bequest be spent in publishing an *Otto Dunkel Memorial Problem Book*, based upon the better problems that have appeared over the years in the AMERICAN MATHEMATICAL MONTHLY. This was a very fitting decision, for Professor Dunkel had served for twenty-eight years as an editor of the MONTHLY Problem Department and is the one chiefly responsible for the Department's growth and high standards. A committee was formed, and in due course the present editors of the Problem Department were asked to assemble the book.

Although we approached the task with enthusiasm, the pressures of present-day college teaching bogged us down, and the assembling of the Problem Book progressed at a discouragingly slow pace. For this we offer our sincerest apologies. We fully realize the shortcomings of certain parts of the book—shortcomings that might not have been so serious had our thinking and work on the project not been so often interrupted for lengthy periods of time.

It was decided that the book should contain a brief biography of Professor Dunkel, a short account of some of the more human aspects of the Problem Department, a selection of the four hundred "best" problems proposed between 1918 (the year in which consecutive numbering of problems was begun, and the year in which Professor Dunkel joined the MONTHLY staff) and 1950, a classification of all problems proposed in this period, and, finally, an index of all these problems so that a reader could locate them and their solutions in the pages of the MONTHLY.

Professor P. R. Rider, a long time friend and colleague of Professor Dunkel, agreed to furnish the biography, and Dean C. W. Trigg, one of the country's noted problemists, agreed to write the human interest section. We deeply thank these men for their very kind assistance.

Since we found it difficult to construct a satisfactory definition of a "good" problem, we decided to secure the composite judgment of a number of mathematicians who, over the years, had shown interest in the MONTHLY Problem Department. These mathematicians sent us their selections of "best" problems, and from these selections we proceeded, largely on the basis of number of votes, to make the final choice. We owe far more to the generous cooperation of our

correspondents in this matter than is implied by the following simple listing of their names:

T. M. Apostol	Fritz Herzog	C. D. Olds
Leon Bankoff	J. B. Kelly	George Piranian
I. A. Barnett	M. S. Klamkin	O. J. Ramler
P. T. Bateman	A. E. Livingston	L. A. Ringenberg
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J. S. Frame	B. H. McCandless	Chih-yi Wang
S. H. Gould	Leo Moser	Albert Wilansky

The chief bugbear of the whole project was the formation of an appropriate classification of all problems proposed between 1918 and 1950, and it is undoubtedly this part of the book that has suffered most from our numerous interruptions. Probably no classification can be entirely satisfactory, but we do hope that the one we finally achieved may be of some use to teachers of courses in college mathematics, to writers of college textbooks, to mathematics clubs and seminars, and to individuals looking for interesting related problem material. The MONTHLY can rightfully boast of possessing one of the finest and most diversified problem departments on the collegiate level. There is evidence to the effect that a large number of research papers and theses for college degrees have had their origin in the MONTHLY Problem Department; it probably would be surprising if we could learn how many. In this connection, the listing of the 251 still unsolved MONTHLY problems for the given period may be of some interest and may stimulate some graduate and undergraduate research. For aid in the tedious task of classifying the nearly three thousand problems, the editors owe much to the expert assistance of Professors D. A. Kearns and S. H. Kimball.

With this we conclude our very inadequate explanation, apology, and thanks.

HOWARD EVES, University of Maine
E. P. STARKE, Rutgers University

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OTTO DUNKEL

P. R. RIDER, Wright-Patterson Air Force Base

Otto Dunkel, son of Frederick W. and Elizabeth Check Dunkel, was born at Richmond, Virginia, May 25, 1869. He died in St. Louis, Missouri, January 15, 1951. He was never married.

Dunkel was forced by circumstances to work during the years when most boys are in college, and it was not until he was twenty-four, in 1893, that he entered the University of Virginia. He received three degrees from that institution, M.E. in 1896, A.B. and A.M. both in 1898. During the academic year 1896–97 he was assistant in astronomy at Virginia under Professor Ormond Stone, founder of the *Annals of Mathematics*. From Virginia he went to Harvard, where he received an A.M. in 1889 and a Ph.D. in 1902. His dissertation, *Regular singular points of a system of homogeneous linear differential equations of the first order*, was written under the direction of Professor Maxime Bôcher.

After receiving his doctor's degree he spent two years as instructor in mathematics at Wesleyan University, Middletown, Connecticut. He then went abroad for two years, studying at the University of Göttingen during 1904–05 and at the University of Paris during 1905–06. Upon returning to the United States he became an instructor at the University of Minnesota, which position he held for one year, then going to an instructorship at the University of Missouri. He was at Missouri during the period 1907–16, except for a leave of absence for the academic year 1909–10, which he spent at the University of Pisa. In 1916 he became assistant professor of mathematics at Washington University, St. Louis. He was promoted to an associate professorship in 1919 and became emeritus in 1939.

Dunkel published numerous papers in various journals, including the *American Mathematical Monthly*, the *Annals of Mathematics*, the *Bulletin of the American Mathematical Society*, the *Proceedings of the American Academy of Arts and Sciences*, *School Science and Mathematics*, and the *Washington University Studies*. With E. R. Hedrick he translated into English Goursat's *Cours d'Analyse Mathématique*.

He served as an editor of the Problem Department of the MONTHLY from 1918 through 1946, being in charge of the department after 1933. He was zealous, conscientious, and very meticulous in this service. He probably worked through every solution that was submitted. If a solution was incorrect in any particular he would explain the matter to the solver and suggest a correction. Often when no solution was forthcoming to an especially difficult problem he supplied it himself. He was quite ingenious, and seemed to be equally at home in all fields of mathematics. He not only served faithfully as an editor, but remembered the Association generously in his will.

Otto Dunkel was a man of very simple tastes and lived quite frugally. He

read a great deal and seemed especially fond of novels in foreign languages, notably French, Spanish, Italian, and German. Although quiet and retiring, he was friendly and was well liked by his colleagues. Because of his wide reading, extensive travel, and a certain dry wit, he was an interesting conversationalist when in the mood.

THE MONTHLY PROBLEM DEPARTMENTS, 1894–1954

C. W. TRIGG, Los Angeles City College

At its inception, the AMERICAN MATHEMATICAL MONTHLY was essentially a problem journal. Conceived and founded by B. F. Finkel at a time when it was difficult to obtain good articles from outstanding mathematicians, the early issues consisted largely of problems—proposed and solved. When Finkel's plan to start the publication began to crystallize, he invited J. M. Colaw to join him. He had never met Colaw, but had been impressed by his contributions, mostly problems, in the *School Visitor*, to which both were avid subscribers.

The editors. Under this joint editorship the MONTHLY was launched with the announced purpose of being "devoted to the solution of problems in pure and applied mathematics, papers on mathematical subjects, biographies of noted mathematicians, etc." The problems were classified under arithmetic, geometry, mechanics, and probability—edited by Finkel; and algebra, calculus, diophantine analysis, and miscellaneous—edited by Colaw. In the first year, the distribution of the problems was arithmetic—41, algebra—43, geometry—36, calculus—33, mechanics—19, diophantine analysis—23, probability—21, and miscellaneous—21. Solutions to 166 of the 237 proposals were published by the end of the year. At first, problems were numbered consecutively in each of the eight sections.

Finkel's interest in problems stemmed from a senseless one which an older half brother heard at the village grocery store and brought home for him to think about. It was: "There is a ball 12 feet in diameter on top of a pole 60 feet high. On the ball stands a man whose eye is 6 feet above the ball. How much ground beneath the ball is invisible to him?" Finkel, 15 years old, was unable to solve the problem with the rules of mensuration which had been made available to him by that time in his small Ohio country school. His teacher indicated that it might be solved by geometry, which Finkel did several years later.

This problem, which later found its way into Finkel's *Mathematical Solution Book*, so impressed Finkel in connection with his own introduction to mathematics that throughout his life he advocated the educational value of a good senseless problem as against that of the so-called practical problem which is so insisted upon by some modern educators. In the introduction to the first issue of the MONTHLY he said: "the solution of problems is one of the lowest forms of mathematical research, . . . yet its educational value cannot be overestimated. It is the ladder by which the mind ascends into higher fields of original research and investigation. Many dormant minds have been aroused into activity through the mastery of a single problem."

In 1903, L. E. Dickson replaced Colaw. In 1904, the arithmetic section was replaced by group theory, and the editorial assistance of Saul Epstein was obtained. In 1905, O. E. Glenn was added to the editorial staff to assist while Finkel was obtaining his Ph.D. Unsolved problems were printed in separate

groups. Multiple solutions were printed in the same and in several issues.

In August 1906, the University of Chicago assumed exclusive control of the MONTHLY with Finkel still editing the problem section. In 1909 the group theory section was abandoned, followed by the miscellaneous section in 1910, and the probability section in 1912. In 1911 the University of Illinois became a co-sponsor with Chicago, and G. E. Wahlin's services were obtained for the number theory (formerly diophantine analysis) section. In 1915 R. P. Baker joined the editorial staff of the problem section, and twelve other universities joined the sponsors.

In 1916 the MONTHLY was taken over by the newly formed Mathematical Association of America. In 1917 the present practice of indexing by problem number and by author was introduced, with bold face type indicating a solution published, italics a solution acknowledged, and ordinary type a problem proposed. The separate sections were abandoned in 1918 and proposals of all types were numbered consecutively beginning with number 2660.

In March 1919 Otto Dunkel joined Finkel as co-editor. The year 1920 was characterized by illuminating discussions and historical notes by Dunkel, R. C. Archibald, and H. S. Uhler. In August 1921, H. P. Manning was added to the staff to assist the other two editors in checking manuscripts for errors and in reading proof. He also contributed notes elaborating the subjects of some problems. This practice was abandoned in 1922.

In 1923 Manning was replaced by Norman Anning, who was replaced in 1924 by H. L. Olson. The present readable style was adopted in 1929.

In 1932 the Board of Governors, believing that the problems proposed were too difficult for most undergraduate college students, decided to split the department into an elementary and an advanced section. W. F. Cheney, Jr., initiated the elementary section in October, the first proposal being numbered E1. The old number sequence was retained in the advanced section.

The elementary section flourished under Cheney's guidance until 1939 when his total duties became so onerous that he requested relief. A fortunate replacement was available in H. S. M. Coxeter, fresh from revising Ball's *Mathematical Recreations*. He carried on until 1945 when the necessity for more research time caused him to be replaced by the present editor, Howard Eves, a keen problemist who was chosen because he had a tidy mind and a good style.

In 1934 Finkel became inactive in the problem department although he remained on the editorial board until his death on February 5, 1947. Dunkel assumed full responsibility of the department. He had a very wide range of interest and of talent in mathematics, and brought to his editorial duties a real scholarship in the subject. His editorial comments were masterful. He read every solution sent to him, and took time to point out oversights to solvers. His advancing age did not seem to impair his abilities. His copy continued to be neat, correct, and adequately marked for printing until the end. It gave Orrin Frink, who replaced Olsen in 1940, but little to do.

In March 1942 bold face titles for the problems were introduced at the suggestion of Editor-in-Chief L. R. Ford. These definitely brightened the appearance of the pages and made solutions easier to identify.

E. P. Starke, former problem editor of *National Mathematics Magazine*, and an ardent contributor to the MONTHLY department, fortunately was available to take over the advanced section in 1947 when Dunkel asked to be relieved. At that time Frink resigned, having convinced the editors that his services were no longer necessary nor desirable for efficient operation.

The MONTHLY has been particularly fortunate in its problem editors. Almost without exception they have had a real enthusiasm for problems, have enjoyed solving them themselves, and have realized the importance of problems in the growth and development of mathematics. They have had wide ranges of interest and mathematical ability and have supplemented the work of their colleagues nicely.

The problems. In the early years of the department the problems ranged from simple grade school exercises to very difficult graduate problems. The first problem printed in the MONTHLY in 1894 was "Arithmetic 1. Reduce 1/3 to 8ths." With the passage of time, the trend has been away from exercises toward problems of more significance.

As Algebra 43, Volume 1, page 433, (1894), F. M. Shields of Cooperwood, Mississippi, proposed a long involved time-rate-distance problem and offered "a city lot at St. Andrews, Florida, to the party sending the EDITOR the first correct answer." G. B. M. Zerr collected the prize.

In 1918, problem 2660, "Prove that the distance measured along the side of a triangle, from the point of contact with the inscribed circle to the point of contact with an escribed circle, is equal to the side of the triangle between the two circles," was given in an examination at Cooper Union, New York City. Twenty-seven students submitted a correct answer.

In 1929, S. A. Corey, Des Moines, Iowa, in proposing 3402 (page 543), offered a prize of $10 to the person sending in a solution or comment most enlightening, in the judgment of the editors. The prize still is uncollected.

Throughout the evolution of the department the various editors have kept well attuned to their readers' tastes and to current mathematical styles as evidenced by their selection of proposals for publication. Problems whose difficulty lies principally in laborious computations have gradually disappeared. Interest in plane and solid geometry problems is currently waning. Problems best solved by table searching are proscribed. More proposals are appearing in abstract algebra and topology. A nice variety in the fields of mathematics, in difficulty of solutions, and in distribution of proposers is now evident in both departments.

Amateurs like the simpler number problems and the lighter recreational ones. Professionals show particular interest in problems connected with the field of analysis.

Some measure of popular interest is given by the number of solutions submitted to a problem. On the basis of this criterion, the three most popular problems have been,

E1111 [1954, 712], which had 70 solvers:

"Our good friend and eminent numerologist, Professor Euclide Paracelso Bombasto Umbugio, has been busily engaged testing on his desk calculator the $81 \cdot 10^9$ possible solutions to the problem of reconstructing the following exact long division in which the digits indiscriminately were replaced by x save in the quotient where they were almost entirely omitted.

$$\begin{array}{rl} & \quad\;\; 8 \\ x\,x\,x\,) & \overline{x\,x\,x\,x\,x\,x\,x\,x} \\ & \quad\;\; x\,x\,x \\ & \quad\overline{} \\ & \quad\;\; x\,x\,x\,x \\ & \quad\;\; x\,x\,x \\ & \quad\overline{} \\ & \quad\;\;\; x\,x\,x\,x \\ & \quad\;\;\; x\,x\,x\,x \\ & \quad\;\;\;\overline{} \end{array}$$

Deflate the Professor! That is, reduce the possibilities to $(81 \cdot 10^9)^0$."

E950 [1951, 566], which had 65 solvers:

"Show that every positive integral power of $\sqrt{2}-1$ is of the form $\sqrt{m}-\sqrt{m-1}$."

E1042 [1953, 420], which had 62 solvers:

"A sequence $\{a_n\}$, where a_0 and a_1 are given numbers, is defined recursively by $a_{n+2}=(a_{n+1}+a_n)/2$. Find $\lim_{n\to\infty} a_n$."

Sources of problems are quite diverse. Many are by-products of research—incidental results which have interest for themselves. Some are constructed around particular relations for test purposes or for publication. Others are submitted without any knowledge that they can be solved, and others are clear examples of plagiarism offered without reference. These are the bothersome ones for the editor, who cannot be expected to recognize all old problems, nor has time to solve all proposals which are submitted without reference or solution.

Any editor will appreciate a proposal couched in non-ambiguous appealing language and accompanied by a solution necessitating facility and ingenuity and by such references as are known. He may publish it and hope for solutions of greater elegance. Also, he may hope that these solutions will not leave too much to be desired in organization, grammatical structure and legibility.

The contributors. During the sixty-one year span 31624 contributions—proposals and solutions—were offered by 3487 individuals. Of these, 1564 were content with a single contribution, 557 with two, 265 with three, 196 with four,

and 116 with five. The contributions of 565 others fell in the 6–25 range, 107 in the 26–50 range, 67 in the 51–100 range, 32 in the 101–200 range, and 13 in the 201–300 range. Five persons made more than 300 contributions.

The most ardent of these was the amazing G. M. B. Zerr who surpassed all rivals in quantity and quality of contributions in each of the 17 years from the inception of the MONTHLY until his untimely death in 1910, to amass the incredible total of 1697 contributions, both proposals and solutions. During this same period, this versatile mathematician, chemical engineer, professor and school administrator contributed with equal fervor to many other contemporary magazines. His closest rival, J. W. Scheffer, totaled 691.

During the first twenty years the number of problems proposed averaged 112 per year. This average dropped off to 79 during the middle period, and increased to 96 during the last twenty years. It was during the latter era that the indefatigable V. Thébault, whose original contributions to mathematical magazines on number properties and the geometry of the triangle and the tetrahedron have been international in scope, addressed his multitudinous contributions to the MONTHLY. The relatively few which the problem editors found room to use raised his total to 582. Following him, principally on the basis of solutions rather than proposals, were current editor E. P. Starke with 523 and C. W. Trigg with 333.

Faithfulness ranks with ardor in measuring the interest of a problem section. In this regard, none can surpass N. A. Court who contributed in each of 40 years and throughout that time encouraged his students to become problemists. Although not such a regular contributor, W. B. Carver's contributions span 56 years. W. E. Buker and B. F. Finkel contributed continuously for 24 years.

In its first year, 1894, the MONTHLY problem department attracted 123 contributors from 30 states. Twenty-four of these were from Ohio. The first foreign contributor was W. F. King of Canada in 1896; the first European, E. Woelffing of Germany in 1900; and the first Asian, T. Hayashi of Japan in 1907.

The number and geographical distribution of contributors varied with the fortunes of the MONTHLY, reaching a low of 46 contributors in 1912. Only 20 states and England were represented in 1911. A steady increase in popularity has been registered since that time, almost directly proportional to the number of subscribers. In 1954, there were 531 contributors from 45 states and territories and 20 foreign countries. The preponderant show of interest fluctuated among Ohio, Illinois, Pennsylvania, and New York until 1922, since when more contributors have lived in New York than in any other state or territory.

During their existence, the problem departments have received contributions from every one of the United States, and from Alaska, District of Columbia, Hawaii, and Puerto Rico; from Algeria, Egypt, and South Africa; from Australia and New Zealand; from Burma, China, India, Japan, Korea, Malaya, Philippine Islands, Singapore S. S., and Thailand; from Austria, Belgium, Czechoslovakia,

Denmark, England, France, Germany, Hungary, Ireland, Luxembourg, Netherlands, Norway, Poland, Portugal, Roumania, Scotland, Spain, Sweden, Switzerland, Turkey, Wales, and Yugoslavia; from British Honduras, Canada, Cuba, Mexico, and Panama; and from Argentina, Brazil, Chile, Uruguay, and Venezuela—a total of 98 political entities.

The problem departments have demonstrated their appeal to persons at all stages of mathematical development—from high school students to retired professors. D. J. Newman sent in his first contribution at age 16 as a student in Stuyvesant High School, New York City, and has kept at it ever since—all through Harvard and as an applied mathematician at Republic Aviation Corporation. R. D. Carmichael, of number theory fame, and many others had their interest in mathematics developed by the problem departments.

Among school teachers who have found an added stimulus in the MONTHLY problems are W. E. Buker, M. L. Constable, P. W. A. Raine, J. Rosenbaum, C. M. Sandwick, and A. Wayne. In the ranks of those eminent in their chosen specialties who were impelled to contribute frequently while graduate students or while actively engaged in university teaching are P. T. Bateman, E. T. Bell, A. A. Bennett, P. Erdös, L. E. Dickson, J. S. Frame, E. Hille, I. Kaplansky, D. H. Lehmer, O. Ore, G. Pólya, G. Szegö, H. S. Vandiver, M. Ward, J. H. M. Wedderburn, and scores more. Many other problemists upon becoming emeritus found continued problem-solving an effective weapon against vegetation, *e.g.*, N. Anning, W. B. Carver, D. L. MacKay, A. Pelletier, W. R. Ransom, and J. B. Reynolds.

Quite a few persons of varying mathematical backgrounds and with other means of livelihood have become problemists to enjoy an avocation wherein they may test their intellectual mettle. Typical enthusiasts have been: dentist L. Bankoff, nurseryman W. B. Clarke, lawyer M. Dernham, ballistics expert M. Goldberg, steel works manager P. Goormatigh, retired telephone engineer W. O. Pennell, automobile dealer P. A. Pizá, insurance inspector V. Thébault, clergyman G. W. Walker, and patent examiner P. Wiernicke.

This volume will come principally to the attention of those interested in problems and hence needing no indoctrination. Upon others we urge consideration of problem constructing and solving as an enjoyable mental exercise, one means of keeping mathematical skills in working order, a stimulus to new lines of investigation, and an ever-new never-boring avocation. If you should find some proposal in a problem department that stimulates you to a solution, send the solution to the editor. It is only in this way that he can know that he is meeting your interests.

Appreciation is hereby expressed for the aid given by previous and present editors in supplying much of the information recorded here.

THE FOUR HUNDRED "BEST" PROBLEMS (1918–1950)

2688 (*Frank Irwin*). With four quantities, a_1, a_2, a_3, a_4, we may, without changing their order, form the following complex fractions:

$$\frac{\frac{a_1}{a_2}}{\frac{a_3}{a_4}} \qquad \frac{a_1}{\frac{a_2}{\frac{a_3}{a_4}}} \qquad \frac{\frac{a_1}{a_2}}{\frac{a_3}{a_4}} \qquad \frac{a_1}{\frac{a_2}{a_3}} \qquad \frac{a_1}{\frac{a_2}{a_3}} \qquad \frac{a_4}{a_4}.$$

But these are not all different values; the first and fourth are equal. Determine how many different rational functions of the quantities $a_1, a_2, \cdots, a_n$ may be obtained in this way, and which can be represented in more than one way as a complex fraction of the above kind, and which in only one way.

2752* (*R. E. Moore*). Test for convergence the series

$$\sum_{n=1}^{\infty} a_n, \qquad a_n = \left[\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n}\right]^2.$$

2758 (*Leonard Richardson*). Prove that, if r be a positive integer,

$$\int_0^{\pi/2} \frac{\sin(2r+1)\psi}{\sin^2\psi}\,d\psi = \frac{\pi}{2},$$

and

$$\int_0^{\pi/2} \frac{\sin 2r\psi}{\sin\psi}\,d\psi = 2\left\{1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots + \frac{(-1)^{r-1}}{2r-1}\right\}.$$

2759 (*J. L. Riley*). Solve the simultaneous functional equations

$$\phi(x+y) = \phi(x) + \frac{\phi(y)\psi(x)}{1-\phi(x)\phi(y)},$$

$$\psi(x+y) = \frac{\psi(x)\psi(y)}{[1-\phi(x)\phi(y)]^2}.$$

2774* (*Frank Irwin*). Evaluate the circulants

$$\begin{vmatrix} 1 & 2 & 3\cdots n-1 & n \\ n & 1 & 2\cdots n-2 & n-1 \\ \cdots & \cdots & \cdots & \cdots \\ 2 & 3 & 4\cdots & n & 1 \end{vmatrix}, \quad \text{and} \quad \begin{vmatrix} a_1 & a_2 & a_3\cdots a_{n-1} & a_n \\ a_n & a_1 & a_2\cdots a_{n-2} & a_{n-1} \\ \cdots & \cdots & \cdots & \cdots \\ a_2 & a_3 & a_4\cdots & a_n & a_1 \end{vmatrix},$$

where, in the latter, $a_1, a_2, \cdots, a_n$ form an arithmetical progression.

* Provided with solution or note by Otto Dunkel. Wherever the asterisk appears hereafter in this section, it will have this meaning.

2784 (*T. H. Gronwall*). Show that all solutions in integers of

$$y^2 = 1 + x + x^2 + x^3 + x^4$$

are given by

$$x = -1, \quad 0, \quad 3;$$
$$y = \pm 1, \ \pm 1, \ \pm 11.$$

2804 (*T. H. Gronwall*). Show that for $|x| < 1$

$$\frac{1}{\sqrt{1-x^4}} \int_0^x \frac{dx}{\sqrt{1-x^4}} = x + \sum_{n=1}^{\infty} \frac{3 \cdot 7 \cdots (4n-1)}{5 \cdot 9 \cdots (4n+1)} x^{4n+1},$$

$$\left(\int_0^x \frac{dx}{\sqrt{1-x^4}} \right)^2 = x^2 + \sum_{n=1}^{\infty} \frac{3 \cdot 7 \cdots (4n-1)}{5 \cdot 9 \cdots (4n+1)} \frac{x^{4n+2}}{2n+1}.$$

2838 (*Lewis Carroll*). A rope is supposed to be hung over a wheel fixed to the roof of a building; at one end of the rope a weight is fixed, which exactly counterbalances a monkey which is hanging on to the other end. Suppose that the monkey begins to climb the rope, what will be the result?

2845* (*E. L. Post*). Prove that if y_x is a solution of the functional equation

$$y_x = \frac{y_{x+1}^2}{x} + y_{x+1}$$

for positive integral values of x with $y_x > 0$, then

$$\lim_{x \to \infty} y_x \log x = 1.$$

2866 (*Norman Anning*). Equilateral triangles whose sides are 1, 3, 5, $\cdots$ are placed so that their bases lie corner to corner in a straight line. Show that the vertices lie upon a parabola and are all at integral distances from its focus.

2871 (*L. G. Weld*). Weight being disregarded, a package may be admitted to the parcel post if the length plus the greatest girth, measured transversely to the length, does not exceed 72 inches. What is the size of the smallest square window through which all admissible rectangular boxes can be passed?

2884 (*E. H. Moore*). Consider an $m \times n$ array A of numbers a_{st} and an $n \times m$ array B of numbers b_{ts}. Show that the system of mn equations:

$$\sum_{tu} a_{st} b_{tu} a_{uv} = 0 \quad (sv),$$

implies the equation:

$$\sum_{ts} a_{st} b_{ts} = 0.$$

The suffixes s, u have the range 1, 2, $\cdots$, m and the suffixes t, v have the range 1, 2, $\cdots$, n.

2892 (*R. T. McGregor*). Two parabolas have parallel axes. Prove that their common chord bisects their common tangent.

2908 (*L. E. Dickson*). If f is a homogeneous polynomial in n variables and H is its Hessian determinant, prove that the Hessian of f^2 is cHf^n, where c is a constant.

2930 (*R. E. Gaines*). If in reducing p/q (p and q integers, $q>p$) to a decimal, the remainder $q-p$ ever appears, then the fraction will give a repeating decimal the number of digits in whose repetend will be exactly twice the number of digits in the quotient already obtained, and the remaining digits may, without further division, be obtained by subtracting the quotient already found from a succession of 9's.

2933* (*H. E. Dudeney*). With ruler and compasses only, divide an equilateral triangle into four rectilinear pieces which may be put together so as to form a square.

2936* (*J. P. Ballantine*). A person drinking from a conical drinking glass tips it at a constant angular rate. At what angle will the delivery be the maximum and at what angle will the surface of the water be a minimum?

2938* (*C. F. Gummer*). If $a, b, \cdots, i$ are real numbers $\geqq 0$, and if

$$D(r) = \begin{vmatrix} a^r & b^r & c^r \\ d^r & e^r & f^r \\ g^r & h^r & i^r \end{vmatrix}$$

is equal to zero for five real values of r other than zero, prove that the determinant $D(1)$ has either two rows or two columns proportional or a single row or column of zeros.

2980* (*J. Rosenbaum*). Locate a point such that the sum of its distances from n given points shall be a minimum.

2991 (*E. J. Oglesby*). Sum the infinite series

$$S_2(x) = 1 + \frac{3x^2}{2!} + \frac{4x^4}{4!} + \frac{6x^6}{6!} + \cdots,$$

where the numerators of the coefficients form a series of numbers whose third differences are all equal to 2.

2994 (*R. M. Mathews*). Can the following construction be made without the use of a regulus? Construct a line which meets four given skew lines.

3011* (*E. T. Bell*). In a certain paper it is stated that "it is easy to prove that, if $p>0$ is an integer, the relation

$$a_1 \sin\frac{\pi}{2p} + a_2 \sin\frac{2\pi}{2p} + \cdots + a_{p-1} \sin\frac{(p-1)\pi}{2p} + a_p = 0$$

necessitates $a_1=a_2= \cdots =a_{p-1}=a_p=0$, the a's being integers." Prove it.

3023 (*E. T. Bell*). The equation $x^p+y^p+z^p=0$ is possible in integers x, y, z prime to the odd prime p, if

$$\frac{1}{2}\left[\frac{1}{2}N_2(p) - \frac{1}{3}N_3(p) + \frac{1}{4}N_4(p) - \cdots + \frac{1}{p-1}N_{p-1}(p)\right] + 1$$

is divisible by p, where $N_r(n)$ is the number of representations (order essential) of n as a sum of r square integers with roots $\neq 0$.

3032* (*Otto Dunkel*). If $a_1, a_2, \cdots, a_n$ are any real or complex quantities which satisfy the equation

$$x^n - na_1x^{n-1} + {}_nC_2a_2^2x^{n-2} + \cdots + (-1)^i {}_nC_ia_i^i x^{n-i} + \cdots + (-1)^n a_n^n = 0$$

where ${}_nC_i=n!/(n-i)!i!$, prove that $a_1=a_2= \cdots =a_n$.

3034 (*J. L. Riley*). If every root of the equation $f'(x)=0$ be subtracted from every root of the equation $f(x)=0$, find the sum of the reciprocals of the differences.

3040 (*William Hoover*). Given the radius, R, of a sphere rolling down two intersecting straight lines including the angle 2α and equally inclined to the horizon; show that the locus of the center of the sphere is an ellipse of semi-axes $R\csc\alpha$, R.

3043 (*O. D. Kellogg*). Let T denote an open continuum of the xy-plane, say the interior of a smooth simple closed curve. Then if U is continuous in T, and is such that

$$\iint_T U\left(\frac{\partial^2 V}{dx^2} + \frac{\partial^2 V}{\partial y^2}\right)dxdy = 0$$

for all functions V with continuous second derivatives in T, and such that (a) the normal derivatives of V vanish on the boundary of T, or, (b) the boundary not being a smooth curve, the first derivatives of V at a point P approach 0 as P approaches any boundary point of T, U is harmonic in T, *i.e.*, U satisfies Laplace's equation $(\partial^2U/\partial x^2)+(\partial^2U/\partial y^2)=0$.

3045* (*S. A. Corey*). If in the equation

$$s!\left[\frac{1}{(s+1)!}+\frac{c_3}{4!(s-1)!}+\frac{c_5}{6!(s-3)!}+\frac{c_7}{8!(s-5)!}+\cdots+\frac{c_s}{(s+1)!2!}\right]=0 \tag{1}$$

$c_3, c_5, c_7, \cdots, c_s$ be given and retain such constant values that (1) is satisfied for all positive odd integral values of s, $(s>1)$, prove that if s be decreased by unity (so that $s=2n$), then the left member will become equal to $\pm B_n$, according as n is odd or even, B_n being Bernoulli's nth number. Also show how any one of the constants c may be found without first finding all the preceding constants.

3051 (*Norman Anning*). Given the sequence: $u_1=2$, $u_2=8$, $u_n=4u_{n-1}-u_{n-2}$, $(n=3, 4, 5, \cdots)$, show that

$$(\pi/12)=\sum_{n=1}^{\infty}\operatorname{arccot} u_n^2.$$

3053 (*J. Lense*). Let $a_1=a$, $a_2=a^{a_1}$, $a_3=a^{a_2}$, $\cdots$, $a_{n+1}=a^{a_n}$, $\cdots$. Discuss $\lim_{n\to\infty}a_n$ as a function of a. (The limit exists also for some values of $a>1$.)

3089* (*Norman Anning*). Given four points, O, A, B, C, on a straight line, to construct, with straightedge only, the point P on the line such that OP shall be the harmonic mean of OA, OB, OC.

3092 (*N. A. Court*). What must be the relations between the coefficients of a cubic equation in order that its roots, considered as lengths, shall form a triangle?

3104* (*Otto Dunkel*). If $f(x)$ is a single-valued and continuous function of x in the interval $a\leqq x\leqq b$ which is not identically zero and which satisfies the inequality $0\leqq f(x)\leqq M$, show that

$$0<\left[\int_a^b f(x)dx\right]^2-\left[\int_a^b f(x)\cos x\,dx\right]^2-\left[\int_a^b f(x)\sin x\,dx\right]^2 \leqq M^2(b-a)^4/12.$$

3118 (*Harry Langman*). If $n>2$, and e is a primitive root of $e^n=1$, show that the determinant $|a_{ij}|$, of order $n-1$ and having the element a_{ij} equal to e^{ij}, has the value

$$(-1)^{(n-1)(n-2)/4}n^{(n-2)/2}.$$

3137 (*Harry Langman*). Show how to draw, using straightedge only, a tangent to the circumference (or an arc) of a circle at a given point, without making use of Pascal's hexagon theorem.

3140 (*C. C. MacDuffee*). Let f be any algebraic form of total degree $m>1$ in n variables, and $H(f)$ its Hessian. Let ϕ be any analytic function. Prove that

$$H[\phi(f)] = H(f)\cdot\left[\left(\frac{\partial\phi}{\partial f}\right)^n + \frac{mf}{m-1}\,\frac{\partial^2\phi}{\partial f^2}\left(\frac{\partial\phi}{\partial f}\right)^{n-1}\right].$$

In particular when $\phi(f)$ is f^r, we have Mrs. Ballantine's generalization of Problem 2908 [1923, 41]. Again, when $\phi(f)=\log f$, we have

$$H(f) = (1-m)f^nH[\log f],$$

which is a generalization of Exercise 22 of Sir Thomas Muir's *Budget of Exercises on Determinants*, this MONTHLY, 1922, p. 10.

3173 (*C. C. Camp*). Two parallel vertical walls stand upon horizontal ground. A ladder of length a has its foot at the bottom of the first wall and leans against the second. A ladder of length b has its foot at the bottom of the second wall and leans against the first. What must be the distance between the walls so that the ladders will cross at height h? When is a solution possible?

3177 (*Samuel Beatty*). If X is a positive irrational number and Y its reciprocal, prove that the sequences

$$(1+X),\quad 2(1+X),\quad 3(1+X),\cdots$$
$$(1+Y),\quad 2(1+Y),\quad 3(1+Y),\cdots$$

contain one and only one number between each pair of consecutive positive integers.

3179* (*Otto Dunkel*). Is the following statement correct?

The intersections of two curves whose equations are given in polar coordinates (ρ, θ) are obtained by solving the two equations simultaneously for ρ and θ.

3207* (*C. N. Mills*). Prove that $\frac{1}{4}m^2\sqrt{3}$ is the maximum area of a triangle which can be formed with the lines a, b, c, subject to the condition that $a^3+b^3+c^3=3m^3$.

3231 (*R. B. Stone*). The root-mean-square of n numbers $x_1, x_2, \cdots, x_n$ is defined by the formula

$$R.M.S. = \left(\frac{x_1^2 + x_2^2 + \cdots + x_n^2}{n}\right)^{1/2}.$$

(a) For what values of n is the $R.M.S.$ of the first n integers also an integer? (b) For what values of a and n is the $R.M.S.$ of the n successive integers, a, $a+1, \cdots, a+n-1$, also an integer? (c) Under what conditions is the $R.M.S.$ of n integers also an integer?

3242 (*R. S. Underwood*). A man finds that a pile of cocoanuts is exactly divisible by n after giving an extra nut to a monkey. He takes away $1/n$th of the remaining nuts and leaves the rest. A second man repeats the process with the rest giving one nut to the monkey and taking away exactly $1/n$th of the rest. This is continued until the nth man leaves at the end a pile of nuts which is exactly divisible by n. How many nuts were there at the beginning and how many at the end?

3249* (*Otto Dunkel*). If $i!f_i=d^if(x)/dx^i$ and

$$D = \begin{vmatrix} f_r & f_{r+1} \cdots f_{r+n-1} \\ f_{r+1} & f_{r+2} \cdots f_{r+n} \\ \cdots & \cdots \\ f_{r+n-1} & f_{r+n} \cdots f_{r+2n-2} \end{vmatrix},$$

prove that $dD/dx=(r+2n-1)D'$, where D' is the determinant D with the subscripts of the last row increased by unity.

3251 (*J. V. Uspensky*). Show that the system

$$\phi(x, y) = x + y + \sum A_{i,k}x^iy^k = 1, \qquad x\phi_x' - ay\phi_y' = 0,$$

where $A_{i,k}>0$, $a>0$, $2\leqq i+k\leqq n$, has one and only one solution in positive numbers.

3252 (*Laenas G. Weld*). Find the equation of the plane curve along which a point, moving with a velocity in constant ratio to its ordinate, will pass from (x_1, y_1) to (x_2, y_2) in the least possible time.

3276 (*L. L. Silverman and J. Tamarkin*). Prove that if ν is an integer greater than or equal to 1, then

$$\int_0^\infty \frac{(1+z)^{-\nu}dz}{\log^2 z + \pi^2} = (-1)^{\nu-1}\int_0^1 \binom{t}{\nu} dt, \qquad \binom{t}{\nu} = \frac{t(t-1)\cdots(t-\nu+1)}{\nu!}.$$

3296 (*J. Rosenbaum*). It is well known that the radius of the inscribed circle of a right triangle is equal to half the difference between the sum of the legs and the hypotenuse. Derive an analogous expression for the radius of the inscribed sphere of a right tetrahedron.

3305 (*Paul Wernicke*). (a) Prove that x^2+y^2, where x and y are integers, cannot be the square of an integer unless x or y is divisible by 3. (b) Modify the theorem for the case that both x and y are divisible by 3. (c) Generalize the theorem so as to cover other exponents, thereby proving a part of Fermat's greater theorem.

3309* (*Otto Dunkel*). If (1, 2, 3, 4, 5, 6) denotes the determinant of the 6th order whose ith row is x_i^2, x_iy_i, y_i^2, x_i, y_i, 1, and if

$$(i, j, k) = \begin{vmatrix} x_i & y_i & 1 \\ x_j & y_j & 1 \\ x_k & y_k & 1 \end{vmatrix},$$

show that

$$(1, 2, 3, 4, 5, 6) = (6, 1, 2)(2, 3, 4)(4, 5, 6)(1, 3, 5) - (1, 2, 3)(3, 4, 5)(5, 6, 1)(2, 4, 6)$$

is an identity in the 12 independent variables x_i, y_i.

3343 (*J. V. Uspensky*). Show that

$$\sum_{n=1}^{\infty} \frac{1}{n} \int_{2n\pi}^{\infty} \frac{\sin z}{z}\, dz = \pi - \frac{\pi}{2} \log 2\pi,$$

and that

$$\sum_{n=1}^{\infty} \frac{1}{n} \int_{n\pi}^{\infty} \frac{\sin z}{z}\, dz = \frac{\pi}{2} - \frac{\pi}{2} \log \pi.$$

3345 (*B. F. Finkel*). A mill wheel of radius a revolves so that its rim has a velocity v, and drops of water are thrown off from the rim. Find the envelope of the paths of the drops.

3348* (*A. C. Aitken*). Show that

$$\frac{\pi}{6} = \sum_{r=1}^{\infty} \operatorname{arccot} 2u_r^2,$$

where $u_r = 4u_{r-1} - u_{r-2}$ with $u_1 = 1$, $u_2 = 3$.

3367 (*Harry Langman*). Given any triangle. On each side construct an equilateral triangle externally. The centers of these triangles determine another equilateral triangle A. Similarly an equilateral triangle B is determined by constructing the equilateral triangle internally. Show that the difference between the areas of the triangles A and B is equal to the area of the given triangle.

3399* (*B. C. Wong*). Prove or disprove

$$\sum_{i=0}^{t} (-1)^i \binom{r+1}{i} \binom{2r-2i}{r} = r + 1,$$

where $t = r/2$ if r is even and $t = (r-1)/2$ if r is odd.

3408 (*J. V. Uspensky*). Show that the integral

$$V_n = \int_0^1 \int_0^1 \cdots \int_0^1 \frac{x_1^2 + x_2^2 + \cdots + x_n^2}{x_1 + x_2 + \cdots + x_n} dx_1 dx_2 \cdots dx_n$$

converges to the limit $2/3$ when n increases indefinitely and that the product $n(V_n - 2/3)$ remains bounded.

3414* (*B. C. Wong*). Prove or disprove

$$\sum_{i=0}^{t} \left[\frac{(r-1)!(r-2i)}{(r-i)!i!} \right]^2 = \frac{(2r-2)!}{r!(r-1)!},$$

where $t = (r-2)/2$ if r is even and $t = (r-1)/2$ if r is odd.

3417 (*C. O. Williamson*). Construct a square so that each side shall pass through a given point.

3418 (*D. H. Dodge*). Prove that

$$\frac{7\sum n^6 + 5(p+1)\sum n^4 + p\sum n^2}{7\sum n^6 - 5(p-1)\sum n^4 - p\sum n^2} = \frac{n^2 + n + p}{n^2 + n - p}.$$

3421* (*Otto Dunkel*). A convex polygon of n sides may be divided into triangles by its diagonals which intersect only at their extremities. Derive an expression for the number of ways in which this may be done.

3430* (*R. M. Sutton*). It is physically possible, given a large number of unit resistances, to make any resistance p/q between two points A and B, where p and q are integers. The result may be accomplished by connecting in parallel q groups of p resistances each, requiring pq resistances. However, it is usually possible to accomplish the same result by a fewer number of unit resistances. The problem is: Find the minimum number of unit resistances necessary to make a resistance p/q between two points A and B in an electric circuit, p and q being both integers.

3438 (*F. P. Matz*). Solve

$$\int_0^{dy/dx} \frac{\cos w \, dw}{16 + 9\sin^2 w} = \frac{1}{12} \tan^{-1} x.$$

3440 (*A. Pelletier*). A triangle is circumscribed about a circle. Prove that the three following lines are concurrent: (1) the line joining the points of contact of any two sides; (2) the line joining the points of intersection of these sides with the bisectors of the opposite angles; (3) the line joining the feet of the altitudes on these sides.

3445 (*Mannis Charosh*). If p is odd and greater than 1, prove that

$$1^2 \cdot 3^2 \cdot 5^2 \cdots (p-2)^2 \equiv (-1)^{(p+1)/2} \bmod p;$$
$$2^2 \cdot 4^2 \cdot 6^2 \cdots (p-1)^2 \equiv (-1)^{(p+1)/2} \bmod p.$$

3448 (*O. D. Kellogg*). Prove that whenever the infinite series with positive terms, $u_1+u_2+u_3+\cdots$, converges, the series

$$u_1/r_1 + u_2/r_2 + u_3/r_3 + \cdots$$

diverges, $r_n=u_n+u_{n+1}+u_{n+2}+\cdots$ being the remainder of the first series after $n-1$ terms.

3459* (*Norman Anning*). It is observed that $3003 = \binom{15}{5} = \binom{14}{6}$. Solve in positive integers the equation

$$\binom{x+1}{y} = \binom{x}{y+1}.$$

3468 (*C. A. Rupp*). Show that the determinant of n^2 elements in the upper left corner of the Pascal triangle

$$\begin{array}{cccc} 1 & 1 & 1 & 1 \cdots \\ 1 & 2 & 3 & \cdots \\ 1 & 3 & \cdots & \\ 1 & \cdots & & \end{array}$$

has the value unity.

3506 (*E. B. Escott*). Solve the n simultaneous equations in n unknowns:

$$\begin{vmatrix} x_1 & x_2 \cdots x_{n-1} \\ x_n & x_1 \cdots x_{n-2} \\ \cdots & \cdots \\ x_3 & x_4 \cdots x_1 \end{vmatrix} = a_1; \qquad \begin{vmatrix} x_2 & x_3 \cdots x_n \\ x_1 & x_2 \cdots x_{n-1} \\ \cdots & \cdots \\ x_4 & x_5 \cdots x_2 \end{vmatrix} = a_2; \cdots$$

$$\begin{vmatrix} x_n & x_1 \cdots x_{n-2} \\ x_{n-1} & x_n \cdots x_{n-3} \\ \cdots & \cdots \\ x_2 & x_3 \cdots x_n \end{vmatrix} = a_n.$$

3514 (*J. P. Ballantine*). Let D_n denote a determinant of order n whose elements are all zeros and ones, and which has 2 ones and $n-2$ zeros in every row and column. Show that (a) for every n, $D_n = \pm 2^m$, where m and n are both even or both odd, or $D_n=0$; (b) if $D_5=0$, then two rows are identical; (c) if two rows of D_n are identical, then two columns are alike, and conversely; (d) $3m \leqq n$;

and (e) property (b) does not hold except for D_2, D_3, and D_5.

3528 (*A. A. Bennett*). If a, b, c be complex numbers such that

$$|a| = |b| = |c| = r \neq 0,$$

then $|(ab+bc+ca)/(a+b+c)| = r$. Generalize.

3536 (*Martin Rosenman*). Consider fractions of the form 1/2, 1/3, 1/4, 1/5, $\cdots$. We seek to determine which n of these fractions (repetitions allowed) give a sum as near unity as possible but actually less than it. Thus for $n=3$, we have $1/2+1/3+1/7=41/42$. Prove or disprove that, in general, the first n of the fractions in the series $1/2+1/3+1/7+1/43+1/1807+\cdots$ give the desired result; in this series each denominator exceeds by 1 the product of all the preceding.

3539 (*R. E. Gaines*). A slender rod of length $2a$ rests on a circular table of radius r, $r>a$. What are the probabilities that neither end, one end, or both ends, will project over the edge of the table?

3547 (*Martin Rosenman*). Consider n points in a plane. Join these in any order to form a closed polygon. Repeat the operation on the n midpoints of the sides of the polygon thus formed, *etc.* Prove that the successive polygons converge to a point.

3552 (*J. M. Feld*). Prove that

$$\begin{vmatrix} e_{11} & 0 & 0 & \cdots & 0 & A_1 \\ e_{21} & e_{22} & 0 & \cdots & 0 & A_2 \\ e_{31} & e_{32} & e_{33} & \cdots & 0 & A_3 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ e_{n1} & e_{n2} & e_{n3} & \cdots & e_{n,n-1} & A_n \end{vmatrix}$$

$$= A_n - \sum A_{n/p_i} + \sum A_{n/p_ip_j} - \cdots + (-1)^s A_{n/p_1p_2\cdots p_s},$$

where (1) $e_{ij}=1$ if j is a divisor of i and $e_{ij}=0$ if j is not a divisor of i; (2) p_s, p_j, $\cdots$, are the distinct prime factors of n.

3565 (*Orrin Frink, Jr.*). Find the ellipse of least area circumscribing a given triangle.

3582 (*D. C. Duncan*). If α and β are positive integers and $\beta>2$ then $2^\alpha+1$ is never divisible by $2^\beta-1$.

3586* (*R. E. Gaines*). If while an ellipse is turned about in its plane it remains tangent to a fixed straight line at a fixed point, its foci trace a curve whose area is $2\pi a(a-b)$.

3591* (*B. F. Kimball*). Let the nth difference of $\log x$ with difference interval

1 be denoted by $\Delta^n \log x$. Show that

$$\lim_{n\to\infty} n^x \log n\, \Delta^n \log x = \Gamma(x).$$

3594* (*H. T. R. Aude*). Find sets of integers for rational right triangles which, as the number increase, approach a 30°—60° right triangle.

3612 (*H. Grossman*). Prove that the ratio of the sum of the hth, $(h+k)$th, $(h+2k)$th, *etc.* coefficients to the sum of all the coefficients in the expansion of $(a+b)^n$ converges to the limit $1/k$, as n approaches ∞, where $h=1, 2, \cdots, k$; and k is a positive integer.

3621 (*A. S. Levens*). Extend the graphical method for the solution of real roots of a quadratic, as given in Dickson's *First Course in the Theory of Equations*, p. 29, to permit the reading of complex roots.

3625* (*R. E. Moritz*). Show that

$$\sum_{k=1}^{n} (-1)^{k+1} {}_nC_k\, k^n = (-1)^{n+1} n!$$

for all positive integral values of n.

3645* (*P. S. Dwyer*). Show that the value of the determinant formed by deleting the kth column from the array

$$\begin{vmatrix}
1 & 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\
0 & {}_1C_1 & {}_2C_1 & {}_3C_1 & {}_4C_1 & \cdots & {}_nC_1 & {}_{n+1}C_1 \\
0 & 0 & {}_2C_2 & {}_3C_2 & {}_4C_2 & \cdots & {}_nC_2 & {}_{n+1}C_2 \\
0 & 0 & 0 & {}_3C_3 & {}_4C_3 & \cdots & {}_nC_3 & {}_{n+1}C_3 \\
0 & 0 & 0 & 0 & {}_4C_4 & \cdots & {}_nC_4 & {}_{n+1}C_4 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\
0 & 0 & 0 & 0 & 0 & \cdots & {}_nC_n & {}_{n+1}C_n
\end{vmatrix}$$

is ${}_{n+1}C_{k-1}$.

3649 (*F. T. O'Doubler*). Solve the functional equation

$$f(xy) = [f(x)]^{\alpha\beta}[f(y)]^{-\beta},$$

where β is a real constant and $f(x)$ is a real continuous and single-valued function of the real variable x.

3652* (*A. F. Stevenson*). The practice of certain cigarette manufacturers of supplying playing cards, poker hands, *etc.* with their cigarette packets, and of offering various articles in exchange for complete sets of these cards, suggests the following problem:

Assuming that each packet of cigarettes contains one of a set of 52 cards,

and that these cards are distributed among the packets at random (the number of packets available being infinite), what is the average minimum number of packets that must be purchased in order to obtain a complete set of cards?

3658* (*J. M. Feld*). The Simson line of a point P on the circumcircle of a triangle ABC is the tangent at the vertex of a parabola tangent to the sides of ABC and having its focus at P.

3666 (*Martin Rosenman*). Set up a one-to-one correspondence between the points in the open interval $0<x<1$ and the points in the closed interval $0\leqq x\leqq 1$.

3667 (*Raphael Robinson*). Show that $(-1)^{n-1}(n-1)2^{n-2}$ is the value of the n-rowed determinant for which $a_{ij}=|i-j|$.

3673 (*E. B. Escott*). Factor $x^8+98x^4y^4+y^8$ into two polynomial factors with integral coefficients.

3674* (*Garrett Birkhoff*). For any positive integer k show that

$$\phi_k=\frac{(2k-2)!}{k!(k-1)!}=\binom{2k-1}{k}\Big/(2k-1)$$

is an integer; and prove the recurrence formula $\phi_n=\sum_{i=1}^{n-1}\phi_i\phi_{n-i}$.

3678 (*B. W. Jones*). Prove the following theorem: If $f(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0=0$, where n is a positive integer, $a_n\neq 0$, and all coefficients are real, has only real roots, then

1. Descartes' Rule of Signs gives exactly the number of positive and the number of negative roots of the equation.
2. $a_r=0$ $(n>r>0)$ implies $a_{r+1}\cdot a_{r-1}<0$.

3682* (*Mannis Charosh*). If a prime p has the form $p=4k+3$ and m is the number of quadratic non-residues less than $p/2$, prove that

(a) $$1\cdot 3\cdot 5\cdots(p-2)\equiv(-1)^{m+k} \bmod p,$$

(b) $$2\cdot 4\cdot 6\cdots(p-1)\equiv(-1)^{m+k+1} \bmod p.$$

3683 (*Raphael Robinson*). Show that the sum of the medians of a simplex of n dimensions is smaller than $2/n$ and greater than $(n+1)/n^2$ times the sum of the edges of the simplex, and that these are the best limits that can be given.

3687* (*Melvin Dresher*). If $S(i,j)$ denotes the sum of the divisors common to i and j, show that:

$$\begin{vmatrix} S(1,1) & S(1,2)\cdots S(1,n)\\ S(2,1) & S(2,2)\cdots S(2,n)\\ \cdots\cdots\cdots\cdots\cdots\\ S(n,1) & S(n,2)\cdots S(n,n)\end{vmatrix}=n!.$$

3690 (*W. M. Whyburn*). Solve the functional equation

$$f(x)f(-x) = c^2 = [f(0)]^2,$$

subject to the single restriction that $f(x)$ be a single-valued, positive, real function of the real variable x.

3692 (*E. P. Starke*). Show that there are four distinct sets of integers which satisfy the equations

$$x_1 + x_2 + x_3 = 54, \qquad x_1^2 + x_2^2 + x_3^2 = 1406.$$

Develop a general method of attack for similar problems in which 54 and 1406 are replaced by a and b.

3696 (*J. B. Reynolds*). A dog directly opposite his master on the banks of a stream, flowing with uniform speed, swims at a still-water speed of two miles per hour heading directly towards his master at all times. The man notes that the dog does not stop drifting down stream until he is two-thirds across measured perpendicularly to the banks, and that it takes five minutes longer to make the trip than if the water had been still. How wide is the stream?

3700 (*J. H. M. Wedderburn*). Find a basis h_{pq} for matrices of order n such that each element of the basis is idempotent and $h_{pq}h_{rs} = k_{pqrs}h_{ps}$, where k_{pqrs} is a rational number.

3703 (*V. Thébault*). Prove that the integral part of the fourth root of the product of eight consecutive integers is equal to x^2+7x+6, where x is the smallest of the eight integers. This result may be used to show that the product of eight consecutive integers is never the fourth power of an integer.

3705* (*Raphael Robinson*). Show that when the quadratic form

$$\sum_1^n |i-j|\, x_i x_j, \qquad n > 1,$$

is reduced to the sum of squares by a real linear transformation, one of the terms will be positive, the other $n-1$, negative.

3707 (*Bernard Friedman*). If p is a prime of the form $3n+1$, it can be expressed as $p = A^2+27B^2$, where A and B are positive integers, if and only if 2 is a cubic residue of p.

3709* (*E. B. Escott*). Determine the values of A in the trinomial $x^{12}+Ax^6y^6+y^{12}$ so that it will have two polynomial factors of the sixth degree with rational coefficients.

3713* (*R. E. Gaines*). Determine the position of a normal chord of a conic which forms a segment of minimum area. Find the area of such a segment of an ellipse.

3716 (*J. D. Hill and H. J. Hamilton*). Let $E_1, E_2, \cdots, E_n, \cdots$ be an infinite sequence of measurable sets in the interval (a, b) such that $mE_n \geqq k > 0$ for $n = 1, 2, \cdots$. Does there necessarily exist some infinite sequence of indices, $1 \leqq r_1 < r_2 < \cdots < r_i < \cdots$ for which the measure of $E_{r_1} \cdot E_{r_2} \cdots E_{r_i} \cdots$ is greater than zero?

3718 (*Frank Morley*). Show that the ellipse through the points given by the complex numbers a, b, c and with center $(a+b+c)/3$ has semi-axes whose lengths are

$$|a + \omega^2 b + \omega c|/3 \pm |a + \omega b + \omega^2 c|/3,$$

where $\omega = (-1+i\sqrt{3})/2$.

3719* (*Morgan Ward*). Prove that

$$\sum_{r=0}^{n} {}_nC_r/(x+r)(x+r+1)\cdots(x+r+n) = 2^n/x(x+2)(x+4)\cdots(x+2n).$$

3720 (*C. J. Coe*). In transforming coordinates from the rectangular system $OX_1Y_1Z_1$ to the congruent rectangular system $OX_2Y_2Z_2$ we have the following table of cosines:

	X_1	Y_1	Z_1
X_2	λ_1	μ_1	ν_1
Y_2	λ_2	μ_2	ν_2
Z_2	λ_3	μ_3	ν_3 ,

the determinant of the array having the value unity. Prove that

$$\lambda_1 + \mu_2 + \nu_3 \geqq -1.$$

3721 (*Albert Whiteman*). Prove by Fermat's method of infinite descent that an odd prime p of the form $3n+2$ has the quadratic non-residue -3.

3726 (*Mannis Charosh*). The vertices of a triangle inscribed in a given circle are the points of tangency of a triangle circumscribed about the circle. Prove that the product of the perpendiculars from any point on the circle to the sides of the inscribed triangle is equal to the product of the perpendiculars from the same point to the sides of the circumscribed triangle.

3731 (*Raphael Robinson*). In how many ways can a_1 1's, a_2 2's, $\cdots$, a_n n's be arranged, so that in reading from the beginning, none of the $(k+1)$'s are reached until at least one of the k's has been reached?

3734 (*A. A. Bennett*). A car with $n (n > 2)$ passengers of different speeds of mental reaction passes through a tunnel and each passenger acquires unconsciously a smudge of soot upon his forehead. Suppose that each passenger

(1) laughs and continues to laugh as soon as and only so long as he sees a

smudge upon the forehead of a fellow passenger;

(2) can see the foreheads of all his fellows;

(3) reasons correctly;

(4) will clean his own forehead when and only when his reasoning forces him to conclude that he has a smudge;

(5) knows that (1), (2), (3), and (4) hold for each of his fellows.

Show that each passenger will eventually wipe his own forehead.

3737* (*J. P. Ballantine*). Derive the following formulas:

(a) $$\pi = \frac{10}{3} - 24\left\{\frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5} - \frac{1}{3\cdot 4\cdot 5\cdot 6\cdot 7} + \frac{1}{5\cdot 6\cdot 7\cdot 8\cdot 9} - \cdots\right\},$$

(b) $$\pi = 3.15 - 360\left\{\frac{1}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8} - \frac{1}{4\cdot 5\cdot 6\cdot 7\cdot 8\cdot 9\cdot 10} + \cdots\right\},$$

(c) $$\log 2 = \frac{17}{24} - 12\left\{\frac{1}{2\cdot 3\cdot 4\cdot 5\cdot 6} - \frac{1}{4\cdot 5\cdot 6\cdot 7\cdot 8} + \cdots\right\},$$

(d) $$\pi = \frac{64}{21} + 96\left\{\frac{1}{1\cdot 3\cdot 5\cdot 7\cdot 9} - \frac{1}{3\cdot 5\cdot 7\cdot 9\cdot 11} + \cdots\right\}.$$

3739 (*Paul Erdös*). Given $n+1$ integers, $a_1, a_2, \cdots, a_{n+1}$, each less than or equal to $2n$, prove that at least one of them is divisible by some other of the set.

3740 (*Paul Erdös*). From a point O inside a given triangle ABC the perpendiculars OP, OQ, OR are drawn to its sides. Prove that

$$OA + OB + OC \geqq 2(OP + OQ + OR).$$

3743 (*Norman Anning*). Two congruent coplanar parabolas have the same line as axis and open in the same direction. Tangents are drawn to the inner from any point of the outer. Prove that the area bounded by the tangents and the arc joining their points of contact is invariant.

3746 (*Paul Erdös*). Given a triangle ABC, with the sides $a>b>c$, and any point O in its interior. Let AO, BO, CO cut the opposite sides in P, Q, R. Prove that

$$OP + OQ + OR < a.$$

3747* (*Frank Irwin*). Find the single condition that all the roots of the secular equation

$$\begin{vmatrix} a_{11}-x & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-x & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-x \end{vmatrix} = 0$$

should be equal, the a's being real and $a_{ji}=a_{ij}$; and hence determine the cases in which all the roots are equal.

3763 (*Paul Erdös*). Given any simple polygon P which is not convex, draw the smallest convex polygon P' which contains P. This convex polygon P' will contain the area P and certain additional areas. Reflect each of these additional areas with respect to the corresponding added side, thus obtaining a new polygon P_1. If P_1 is not convex, repeat the process, obtaining a polygon P_2. Prove that after a finite number of such steps a polygon P_n will be obtained which will be convex.

3764 (*J. S. Frame*). Prove that

$$\sinh^{n+1} nx + \cosh^{n+1} nx < \cosh^n (n+1)x$$

for all real positive values of x, and for all real values of n greater than unity. Show that the inequality is reversed when $0<n<1$ and $x>0$; whereas equality holds either if $n=0$ or 1, or if $x=0$.

3766* (*M. E. Levenson*). Evaluate

$$\int_0^\infty e^{-x} \log^2 x \, dx.$$

3771 (*Hansraj Gupta*). Prove that

$$61! + 1 \equiv 0 \bmod 71, \qquad 63! + 1 \equiv 0 \bmod 71.$$

Prove the general result of which these are particular cases.

3776 (*E. P. Starke*). Determine all triangles whose sides are relatively prime integers and such that one angle is double another.

3778 (*L. J. Adams*). Solve

$$\frac{dy}{dx} = \frac{1}{x} + \frac{1}{y}.$$

3780 (*J. M. Feld*). In triangle $A_1A_2A_3$ the transversal A_iD_i divides A_jA_k in the ratio $A_jD_i:D_iA_k=p_i:q_i$, where ijk is a cyclic permutation of 123. The transversals A_iD_i and A_jD_j intersect in P_k. Find the value of the cross ratio

$$\frac{P_3P_2}{P_2A_1} \Big/ \frac{P_3D_1}{D_1A_1}$$

in terms of the p's and q's. Show that Ceva's theorem is a special case.

3783 (*J. Barinaga*). Show that

$$\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2^2 & 3^2 & \cdots & n^2 \\ 1 & 2^3 & 3^3 & \cdots & n^3 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 2^n & 3^n & \cdots & n^n \end{vmatrix}$$

$$= 1!2!3!\cdots n!\left[\binom{n}{1} - \frac{1}{2}\binom{n}{2} + \frac{1}{3}\binom{n}{3} - \cdots + (-1)^{n+1}\frac{1}{n}\binom{n}{n}\right].$$

3784 (*J. M. Feld*). Prove that for any positive integer k the following determinant is zero.

$$\begin{vmatrix} \frac{1}{2!} & \frac{1}{3!} & \frac{1}{4!} & \cdots & \frac{1}{(2k+1)!} & \frac{1}{(2k+2)!} \\ 1 & \frac{1}{2!} & \frac{1}{3!} & \cdots & \frac{1}{(2k)!} & \frac{1}{(2k+1)!} \\ 0 & 1 & \frac{1}{2!} & \cdots & \frac{1}{(2k-1)!} & \frac{1}{(2k)!} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \frac{1}{2!} & \frac{1}{3!} \\ 0 & 0 & 0 & \cdots & 1 & \frac{1}{2!} \end{vmatrix}.$$

3792 (*F. Morley*). A square is divided into n^2 unit squares, like a chessboard. Any two horizontal lines and any two vertical lines form a rectangle. We count a square as a rectangle. Each rectangle has a breadth b, less than or equal to its length. There is one rectangle of breadth n, namely the original square. Prove that there are 2^3 rectangles of breadth $n-1$, 3^3 of breadth $n-2$, $\cdots$, n^3 of breadth 1.

Deduce the formula

$$1^3 + 2^3 + \cdots + n^3 = [n(n+1)/2]^2.$$

3801 (*D. H. Lehmer*). Show that

$$\operatorname{arccot} 1 = \operatorname{arccot} 2 + \operatorname{arccot} 5 + \operatorname{arccot} 13 + \operatorname{arccot} 34 + \cdots,$$

where these integers constitute every other term of the Fibonacci series and satisfy the recurrence $u_{n+1} = 3u_n - u_{n-1}$.

3802 (*D. H. Lehmer*). Let $0, u_1, u_2, u_3, \cdots$ be a sequence of numbers satisfying the recurrence $u_{n+1}=au_n+bu_{n-1}$, and consider the function

$$f(x) = \sum_{n=1}^{\infty} u_n x^n/n!.$$

Show that $f(x)=-e^{ax}f(-x)$.

3805* (*R. E. Gaines*). Determine a point P on $b^2x^2-a^2y^2=a^2b^2$ such that the tangent and normal lines at that point shall be normal and tangent respectively to $b^2x^2-a^2y^2=-a^2b^2$, and hence, that if the hyperbola and its conjugate be together considered as a single curve, $(b^2x^2-a^2y^2)^2-a^4b^4=0$, a rectangle may be drawn which is both an inscribed and a circumscribed figure.

3810 (*Oystein Ore*). One may define a new "multiplication" in the system of all positive real numbers by putting

$$[a, b] = a^b.$$

Determine all positive *rational* numbers for which this multiplication is:

I) *Commutative* $[a, b]=[b, a]$.

II) *Associative* $[a, [b, c]]=[[a, b], c]$.

III) *Right-hand or left-hand distributive*

$$[(a + b), c] = [a, c] + [b, c], \qquad [c, (a + b)] = [c, a] + [c, b].$$

3814 (*I. J. Schoenberg*). Let $f(t)$ be a one-valued and complex-valued function which satisfies the functional relation

$$e^{iks}f(t) = f(t + s) - f(s) \tag{1}$$

for all real values of t and s, where k is a real constant $\neq 0$. Prove that

$$f(t) = C(e^{ikt} - 1)/ik \qquad (C \text{ a constant}). \tag{2}$$

Remark. It should be noticed that for $k=0$, (1) and (2) reduce to (1′) $f(t+s)=f(t)+f(s)$ and (2′) $f(t)=Ct$ respectively. It is known that (1′) implies (2′) only if some additional assumption is made on $f(t)$, for instance, boundedness in the neighborhood of some point (see G. Hamel, *Mathematische Annalen*, vol. 60, 1905, pp. 459–462). No such assumption is needed if $k\neq 0$.

3815* (*P. Turán*). Given the function of a real variable x

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{x}{4^n};$$

show that there is a positive constant c_1, independent of x, such that

$$|f(x)| < c_1 \log \log x, \qquad x > e. \tag{1}$$

Show also that there exists a sequence $x_1<x_2< \cdots \to \infty$, and a positive constant c_2, independent of x, such that

(2) $$|f(x_\nu)| > c_2 \log\log x, \qquad \nu = 1, 2, \cdots.$$

3816 (*E. Weiszfeld*). Given the function of the complex variable z

$$f(z) = \sum_{i=1}^{n} \frac{a_i - z}{|a_i - z|},$$

where a_i, $(i=1, 2, \cdots, n)$, denotes any given complex number; show that $|f(z)|$ takes on its maximum value in any domain not containing the points a_i at the boundary of the domain.

3820 (*Paul Erdös*). Let $a_1<a_2< \cdots <a_n<2n$ be positive integers such that no one of them is divisible by any other member of the sequence; then $a_1 \geqq 2^k$, where k is defined by the inequalities $3^k<2n<3^{k+1}$. This estimate for a_1 is the best possible.

3824 (*F. A. Lewis*). Determine the roots of the characteristic equation of the matrix

$$V = (v_{rc}) = (\epsilon^{(r-1)(c-1)}), \quad \text{where } \epsilon = e^{2\pi i/n}.$$

3826 (*W. Macray*). Solve the system of partial differential equations,

$$U\frac{\partial W}{\partial x} + 2W\frac{\partial U}{\partial x} = 0, \qquad U\frac{\partial W}{\partial x} + 2W\frac{\partial V}{\partial y} = 0, \qquad \frac{\partial U}{\partial y} - \frac{\partial V}{\partial x} = 0.$$

3829 (*J. D. Hill*). Let C be a simple closed rectifiable plane curve and P an arbitrary point inside of C. (a) Show that there exist two points A and B on C such that P bisects the chord AB. (b) Does this property remain true if the curve is non-rectifiable?

3834 (*Paul Erdös*). Let $a_1<a_2< \cdots <a_n \leqq 2n$ be a sequence of positive integers. Then

$$\min\,[a_i, a_j] < 6([n/2] + 1),$$

where $[a_i, a_j]$ denotes the least common multiple of a_i and a_j. This is the best possible estimate.

3835 (*Paul Erdös*). Let $a_1<a_2< \cdots <a_n<2n$ be a sequence of positive integers. Then

$$\max\,(a_i, a_j) > \frac{38n}{147} - c,$$

where c is independent of n, and (a_i, a_j) denotes the greatest common divisor of a_i and a_j. This is the best possible estimate.

3863* (*S. B. Townes*). The vertices of a simplex in n dimensions are O, P_1, P_2, $\cdots$, P_n. Let $OP_i=(a_{ii})^{1/2}$ and the cosine of the angle between OP_i and OP_j be $a_{ij}/(a_{ii}a_{jj})^{1/2}$, $(i, j=1, 2, \cdots, n)$. Show that r_n, the radius of the circum-

scribed hypersphere, is given by

$$r_n^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ii} a_{jj} A_{ij} / 4 \left| a_{ij} \right| ,$$

where A_{ij} is the cofactor of a_{ij} in the symmetric determinant $|a_{ij}|$.

3868 (*Arnold Dresden*). Prove that, if $0 \leqq \alpha \leqq x_1 \leqq x_2$ and n is a positive integer, then

$$x_2^{1/n} - x_1^{1/n} \leqq (x_2 - \alpha)^{1/n} - (x_1 - \alpha)^{1/n}.$$

3872 (*E. R. Ott*). One man has m coins and another has n. They match coins until one player has won all the coins. Find the average number of tosses required to end the game.

3879 (*Victor Thébault*). Show that it is possible to determine a plane section, limited by three faces of a tetrahedron which divides both the surface and the volume into equal parts. Show that the plane passes through the center of the inscribed sphere.

3885 (*Victor Thébault*). The product of n consecutive positive integers, n being odd, is divisible by their sum, except in the case where, n being prime, the arithmetic mean of the n integers is divisible by n. Examine the case where n is even.

3893 (*Norman Anning*). From the vertices of a regular n-gon three are chosen to be the vertices of a triangle. Prove that the number of essentially different possible triangles is the integer nearest to $n^2/12$.

3903 (*Simon Mowshowitz*). Denote $[(n!)!]!$ by $n(!)^2$, *etc.*, $n(!)^0 = n$. Prove that for $k \geqq 2$,

$$\frac{n(!)^k}{(n!)^{[n-1]![n!-1]![n(!)^2-1]!\cdots[n(!)^{k-2}-1]!}}$$

is an integer.

3907 (*V. W. Graham*). Find all the roots of the equation

$$\frac{(x^2 - x + 1)^3}{x^2(x-1)^2} = \frac{(a^2 - a + 1)^3}{a^2(a-1)^2} .$$

3909 (*Béla Sz.-Nagy*). Let $P_1, P_2, \cdots, P_{n+2}$ be $n+2$ points in n-dimensional space, $n \geqq 2$, no three of the points on the same straight line. Let the symbol $[P_{i_1} P_{i_2} \cdots P_{i_s}]$ denote the least convex polyhedron containing in its interior the points indicated. Show that if $n = 2, 3$, then there exist always subscripts i, k, such that $[P_i P_k]$ is not an edge of $[P_1 P_2 \cdots P_{n+2}]$. Show that if

$n>3$, then this statement does not remain true.

3914 (*W. B. Campbell*). A dog is tied to a rope of length L, which is fastened on the other side of a smooth topped fence of height h at a point a units from the top, $L>h+a$. Discuss the shape and dimensions of the region over which he can roam; and find its area.

3918* (*B. M. Stewart*). Given a block in which are fixed k pegs and a set of n washers, no two alike in size, and arranged on one peg so that no washer is above a smaller washer. What is the minimum number of moves in which the n washers can be placed on another peg, if the washers must be moved one at a time, subject always to the condition that no washer be placed above a smaller washer?

3919 (*Richard Bellman*). Prove that

$$\begin{vmatrix} \frac{x}{1-x} & 1 & 0 & 0 & 0 \cdots & 0 \\ \frac{x^2}{1-x^2} & \frac{x}{1-x} & 2 & 0 & 0 \cdots & 0 \\ \frac{x^3}{1-x^3} & \frac{x^2}{1-x^2} & \frac{x}{1-x} & 3 & 0 \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{x^r}{1-x^r} & \frac{x^{r-1}}{1-x^{r-1}} & \cdots & \cdots & \cdots & \frac{x}{1-x} \end{vmatrix} = \frac{r!x^{r(r+1)/2}}{(1-x)(1-x^2)\cdots(1-x^r)}.$$

3923 (*R. E. Gaines*). It is known that the circumcircle of the triangle formed by three tangents to a parabola passes through the focus. Show that the diameter d of the circle is given by $d \sin\alpha \sin\beta \sin\gamma = a$, where α, β, γ are the angles which the tangents make with the axis of the parabola, $y^2=4ax$.

3939* (*J. H. Curtiss*). Define the function $f(x)$ by the relations

$$f(x) = x \sin(1/x), \qquad x > 0,$$
$$= 0, \qquad x = 0.$$

Show that $|f(x_1)-f(x_0)|/|x_1-x_0|^\alpha$ is bounded for $0 \leqq x_0 \leqq 1$, $0 \leqq x_1 \leqq 1$, if and only if $\alpha \leqq 1/2$.

3942 (*H. E. Tester*). A man is standing at the junction of two perpendicular crossroads, and his dog, at a distance a from the junction along one of the roads, is watching him. At a given instant the man starts to walk with speed v along the other road, and the dog to run directly towards his master with speed $2v$. Determine the curve of pursuit.

3954* (*Oystein Ore*). From three elements a, b, c in given order one can form two products, namely $(ab)c$ and $a(bc)$ when the associative law is not assumed. Similarly four elements a, b, c, d give $N_4=5$ products; $[(ab)c]d$, $[a(bc)]d$, $a[(bc)d]$, $a[b(cd)]$, $(ab)(cd)$. Find the general expression for the number N_i of products with i factors.

3957* (*Otto Dunkel*). Given a triangle ABC with angles $A<B<C$, show that there are precisely one, two, three straight line segments which bisect both its perimeter and area according as

$$1-\frac{\sin A}{\sin B}\gtreqless 2\tan^2(A/2)\tan^2(B/2),$$

where we may replace B by C. If $B=C$, there are one, two, three such segments, according as $A\gtreqless A_0$, where $\sin(A_0/2)=\sqrt{2}-1$.

3965* (*H. S. Wall*). Show that for a real or complex x, $|x|\leqq 1$,

$$\frac{|x|}{1+|x|}\leqq|\log(1+x)|\leqq\frac{|x|(1+|x|)}{|1+x|}.$$

3967 (*V. Thébault*). For a given triangle ABC a second triangle $A'B'C'$ is formed, where AA', BB', CC' are segments of altitudes and $AA'/BC=BB'/CA=CC'/AB=k$. (1) Show that the two triangles have the same centroid. (2) Examine the variation of the area of $A'B'C'$. (3) For what value of k do the two triangles have the same angle of Brocard? (4) If $k=\pm 1$, show that the centers of squares constructed exteriorly, or interiorly, on the sides of $A'B'C'$ are the vertices of ABC.

3980 (*Esther Szekeres*). The symmetric polynomials $y_1, y_2, \cdots, y_n$ in the variables $x_1, x_2, \cdots, x_n$ are of the degrees indicated by the subscripts, and are algebraically independent. If $f(x_1, x_2, \cdots, x_n)$ is any given polynomial symmetric in the x's, show that it can be expressed as a polynomial in the y's.

3990 (*V. Thébault*). Let A', B', C' be the centers of squares $BCA_1'A_2'$, $CAB_1'B_2'$, $ABC_1'C_2'$ constructed interiorly on the sides of triangle ABC with the centroid G and the angle V of Brocard. If $\cot V=7/4$, show that: (1) The centers A'', B'', C'' of the squares constructed interiorly on the sides of $A'B'C'$ lie on a straight line through G. (2) The angle V' of Brocard of $A'B'C'$ is such that $\cot V'=2$. (3) The straight lines joining A, B, C respectively to the midpoints of $A_1'A_2'$, $B_1'B_2'$, $C_1'C_2'$ are parallel. (4) The distance of the circumcenter from the orthocenter of the orthic triangle is equal to one fourth of the perimeter of the last triangle.

3996 (*E. H. Clarke*). Sum the series

$$\sum_{n=1}^{\infty}[(n-1)k]!/(nk)!,$$

where k is any integer greater than unity.

3999* (*G. B. Van Schaack*). Let $f(x)$ be a polynomial of degree n with distinct real roots x_i, $(i=1, 2, \cdots, n)$. Let λ_i be the reciprocal of the slope of the curve $y=f(x)$ at $x=x_i$. Let ρ_j, $(j=1, 2, \cdots, n-1)$, be the algebraic radius of curvature of the curve at the critical point of the curve which lies between x_j and x_{j+1}. (a) Show that if $n>1$, then $\lambda_1+\lambda_2+\cdots+\lambda_n=0$. (b) Show that if $n>2$, then $\rho_1+\rho_2+\cdots+\rho_{n-1}=0$.

4002 (*F. A. Lewis*). Give an interpretation to the function that results from the Euler ϕ-function when the minus signs are changed to plus, namely $f(n)=n(1+1/p_1)(1+1/p_2)\cdots(1+1/p_k)$.

4009 (*J. H. M. Wedderburn*). If the roots of $x^n-c_1x^{n-1}+\cdots+(-1)^nc_n$ are the variables $x_1, x_2, \cdots, x_n$, find the Jacobian of $c_n, c_{n-1}, \cdots, c_1$ with respect to $x_1, x_2, \cdots, x_n$.

4012* (*V. Thébault*). Find a number of n digits $N_0=a_1a_2\cdots a_n$, $(a_1\neq 0)$. such that if we transpose the first k digits from left to right, $(k=1, 2, 3, \cdots, n-1)$, the $n-1$ numbers thus obtained $N_1=a_2a_3\cdots a_na_1$, $N_2=a_3a_4\cdots a_na_1a_2$, $\cdots$, $N_{n-1}=a_na_1a_2\cdots a_{n-1}$ are each multiples of N_0.

4014* (*P. Erdös*). Show that, if S_1 and S_2 are two squares contained in the unit square so that they have no point in common, the sum of their sides is less than unity.

It is very likely true that, if we have k^2+1 squares contained in the unit square so that no two of them have a point in common, the sum of their sides is less than k.

4019 (*Robin Robinson*). Given a triangle ABC. Prove that the bisectors of the interior and exterior angles at C, the side AB and its perpendicular bisector, and the perpendiculars to AC at A and to BC at B, are all tangent to a parabola. Locate its focus.

4021 (*Orrin Frink, Jr.*). The differential operator D^2+1 may be factored in many ways; for example it may be written $(D+\cot x)(D-\cot x)$, or $(\sec x\cdot D)(\cos x\cdot D+\sin x)$, or $(\sin x\cdot D+2\cos x)(D\csc x)$. Show that the most general method of expressing the differential operator $(D+a)^2+b^2$ as the product of two real first order differential operators is given by the formula

$$(D+a)^2+b^2=r^{-1}\left[D+a-b\tan(bx+c)-r'/r\right]\cdot r\left[D+a+b\tan(bx+c)\right],$$

where a, b, and c are real numbers, and r is a differentiable function of x.

4023 (*J. A. Greenwood*). Find an expression for the determinant of order $2n$

$$\begin{vmatrix} \theta I_n & A_n \\ A_n & \theta I_n \end{vmatrix},$$

where θI_n is a square matrix of order n having the variable θ in the principal diagonal and zeros elsewhere, and A_n is a square symmetric matrix of order n with a for each principal diagonal element, unity for the elements in the two parallels immediately above and below this principal diagonal, and zeros elsewhere.

4042 (*Henry Scheffé*). Prove that if A is a fixed positive definite hermitian matrix and X is a variable non-negative hermitian matrix (rank = index), then the minimum value of the determinant $|A+X|$ is $|A|$ and is attained if and only if $X=0$.

4045 (*A. M. Glicksman*). Show that γ, Euler's constant, is given by

$$\gamma = \sum_{r=2}^{\infty} (-1)^r \frac{g_r}{r}, \qquad g_r = \sum_{k=1}^{\infty} \frac{1}{k^r}, \qquad \gamma = \lim_{n\to\infty} \left(\sum_{i=1}^{n} \frac{1}{i} - \log n \right).$$

4046* (*Otto Dunkel*). Show that γ, Euler's constant, is given by

$$\gamma = 2\left[1 - \log 2 - \frac{\tau_3}{3} - \frac{\tau_5}{5} - \cdots\right], \qquad \tau_r = \sum_{i=1}^{\infty} \frac{1}{(2i+1)^r}.$$

4047 (*T. R. Running*). Triangles have the sides $x-1$, x, $x+1$, the altitude h with x as base, and area A, where x, h, A are whole numbers. The first six possible triangles are given by the table

n	h	x	A
0	0	2	0
1	3	4	6
2	12	14	84
3	45	52	1170
4	168	194	16296
5	627	724	226974

Do the relations

$$h_{n+2} = 4h_{n+1} - h_n, \qquad x_{n+2} = 4x_{n+1} - x_n, \qquad A_{n+2} = 14A_{n+1} - A_n,$$

hold for all the triangles fulfilling the given conditions?

4050 (*Arnold Dresden*). If $a_1, a_2, \cdots, a_n$ are n distinct complex numbers, $n>1$, such that no two differ by a multiple of π, prove that

$$\sum_{k=1}^{n} \prod_{i=1, i\neq k}^{n} \cot(a_k - a_i) = \sin\frac{n\pi}{2}.$$

4051 (*Arnold Dresden*). If $a_1, a_2, \cdots, a_n$ are n distinct complex numbers, $n>1$, such that none of them and none of the differences is a multiple of π, show that

$$\sum_{j=1}^{n} \cot a_j \prod_{i=1, i\neq j}^{n} \cot (a_j - a_i) + (-1)^n \prod_{i=1}^{n} \cot a_i = \sin \frac{(n+1)\pi}{2}.$$

4053* (*E. P. Starke*). Show that all triangles inscribed in an ellipse and having their centroids at the center of the ellipse have the same area, which is the greatest possible area for an inscribed triangle.

Show that all triangles circumscribed about an ellipse and having their centroids at the center of the ellipse have the same area, which is the least possible area for a circumscribed triangle.

4054* (*V. Thébault*). Find the base less than 100 for which the number 2101 is a perfect square.

4065* (*P. Erdös*). (1) Let n given points have the property that the straight line joining any two of them passes through a third point of the set. Show that the n points lie on a straight line.

(2) Given n points which do not all lie on the same straight line, prove that if we join every two of them we obtain at least n distinct straight lines.

4066 (*Richard Bellman*). Prove that

$$\int_0^\infty \frac{dx}{\Gamma(x)} = \int_0^1 \left[1 + \frac{e}{x} - \frac{e}{1!(x+1)} + \frac{e}{2!(x+2)} - \cdots\right] \frac{dx}{\Gamma(x)}.$$

4067 (*H. S. Wall*). Prove that, if $0<x<1$,

$$\prod_{n=1}^{\infty} (1 - x^{2n-1}) = 1 \Big/ \left[1 + \sum_{k=1}^{\infty} x^{k(k+1)/2}/(1-x)(1-x^2)(1-x^3)\cdots(1-x^k)\right].$$

4070 (*P. Erdös*). Let ρ denote the length of the radius of the inscribed circle of the triangle ABC, let r denote the circumradius and let m denote the length of the longest altitude. Show that $\rho + r \leqq m$.

Correction. The proposer intended to exclude obtuse-angled triangles.

4072 (*Richard Bellman*). Show that

$$e^x = \frac{(1-x^2)^{1/2}(1-x^3)^{1/3}(1-x^5)^{1/5}\cdots}{(1-x)(1-x^6)^{1/6}(1-x^{10})^{1/10}\cdots}, \qquad |x| < 1,$$

where the exponents in the numerator are integers with an odd number of unrepeated prime factors; and those in the denominator have an even number of unrepeated prime factors.

4083 (*P. Erdös*). Let $a_1<a_2<\cdots<a_x\leqq n$ be an arbitrary sequence of positive integers such that no a_i divides the product of the others, then $x\leqq\pi(n)$,

where $\pi(n)$ denotes the number of primes not exceeding n.

4085 (*V. Thébault*). Given an equilateral hyperbola (H) and a circle (O) passing through the center ω of (H), show that the necessary and sufficient condition for the existence of an infinite number of triangles inscribed in (H) and circumscribing the circle is that the center O of the circle lies on (H). Consider the envelope of the sides of these triangles.

4086 (*P. Erdös*). Let $A_1, A_2, \cdots, A_{2n+1}$ be the vertices of a regular polygon and let O be any point in its interior. Show that at least one of the angles A_iOA_j satisfies the relation:

$$\pi\left(1-\frac{1}{2n+1}\right) \leqq A_iOA_j \leqq \pi.$$

4091 (*Morgan Ward*). Given the three series

$$z-\frac{z^5}{2\cdot4\cdot5}+\frac{z^9}{2\cdot4\cdot6\cdot8\cdot9}-\frac{z^{13}}{2\cdot4\cdot6\cdot8\cdot10\cdot12\cdot13}+\cdots,$$

$$\frac{z^3}{2\cdot3}-\frac{z^7}{2\cdot4\cdot6\cdot7}+\frac{z^{11}}{2\cdot4\cdot6\cdot8\cdot10\cdot11}-\frac{z^{15}}{2\cdot4\cdot6\cdot8\cdot10\cdot12\cdot14\cdot15}+\cdots,$$

$$\frac{z^2}{1\cdot2}-\frac{z^6}{1\cdot3\cdot5\cdot6}+\frac{z^{10}}{1\cdot3\cdot5\cdot7\cdot9\cdot10}-\frac{z^{14}}{1\cdot3\cdot5\cdot7\cdot9\cdot11\cdot13\cdot14}+\cdots,$$

prove that the sum of the squares of the first two series is double the third series.

4093 (*B. M. Stewart*). A store conducted the following game in which the player paid 10¢, and if he won received \$1 in merchandise. Out of five throws, each time throwing ten dice, a win was declared if the player had fourteen or more appearances of the numbered face named by him before playing; the player had the additional advantage that on the first throw of ten dice he could count the face occurring the greatest number of times as if it were the special face he had selected. The problem is to compare the theoretical probability with the odds offered by the store.

4101 (*R. C. Buck*). Show that

$$(n) = \begin{vmatrix} (1,1)^\lambda & (1,2)^\lambda \cdots (1,n)^\lambda \\ (2,1)^\lambda & (2,2)^\lambda \cdots (2,n)^\lambda \\ \cdots\cdots\cdots\cdots\cdots \\ (n,1)^\lambda & (n,2)^\lambda \cdots (n,n)^\lambda \end{vmatrix}$$

$$= (n!)^\lambda\left(1-\frac{1}{2^\lambda}\right)^{[n/2]}\left(1-\frac{1}{3^\lambda}\right)^{[n/3]}\left(1-\frac{1}{5^\lambda}\right)^{[n/5]}\cdots,$$

where (i, j) means the greatest common divisor of the integers i, j.

4104 (*E. T. Bell*). Two symmetric functions, $M(x_1, \cdots, x_n)$, $S(x_1, \cdots, x_n)$, of n non-negative integers $x_1, \cdots, x_n$ are defined as follows

$$M(x_1, \cdots, x_n) \equiv M'(x_1) \cdots M'(x_n),$$

in which $M'(x) = 1, -1, 0$, according as $x = 0$, $x = 1$, $x > 1$; if $S_j(x_1, \cdots, x_n)$ is the jth elementary symmetric function of $x_1, \cdots, x_n$,

$$S(x_1, \cdots, x_n) \equiv 1 + \sum_{j=1}^{n} jS_j(x_1, \cdots, x_n).$$

Prove that $\sum M(x_1 - b_1, \cdots, x_n - b_n)S(b_1, \cdots, b_n)$ equals the number of positive integers in the set $x_1, \cdots, x_n$; and equals unity if $x_1 = x_2 = \cdots = x_n = 0$, where the summation refers to all integers b_i such that $0 \leqq b_i \leqq x_i$, $(i = 1, 2, \cdots, n)$.

4108* (*G. Pólya*). Let ${}_nP_r$ be the number of those permutations of n elements which are the products of exactly r cycles without common elements. For instance, ${}_4P_2 = 11$. Let ${}_nQ_r$ be the number of different classifications of n distinct elements into exactly r classes. For instance, ${}_4Q_2 = 7$. Prove that

(1) $${}_nP_1x + {}_nP_2x^2 + \cdots + {}_nP_nx^n = x(1 + x)(2 + x) \cdots (n - 1 + x),$$

(2) $${}_nQ_1x + {}_nQ_2x(x - 1) + \cdots + {}_nQ_nx(x - 1) \cdots (x - n + 1) = x^n.$$

4118* (*Otto Dunkel*). Show that

$$\sum_{t=0}^{n} (-1)^{n+t} \frac{t^{n+4}}{t!(n-t)!} = \frac{(n+4)(n+3) \cdots n}{6!8} [15n^3 + 30n^2 + 5n - 2], \quad n \geqq 0,$$

and that each member of this equality is a non-negative integer. If n is a negative integer, the right member is an integer; what meaning may be given to the result in this case?

4124 (*T. W. Anderson, Jr.*). Consider the set of n by n matrices whose entries are positive integers or zero. Let the sum of the entries in the ith row be r_i, $i = 1, 2, \cdots, n$, and the sum of the entries in the jth column be c_j, $j = 1, 2, \cdots, n$. For specified r_i and c_j, positive integers or zero, with

$$\sum_{i=1}^{n} r_i = \sum_{j=1}^{n} c_j,$$

what are the minimum and maximum sums of entries in the main diagonal, *i.e.*, the minimum and maximum traces?

4132 (*T. H. Matthews*). If an aircraft travels at a constant air speed, and traverses (with respect to the ground) a closed curve in a horizontal plane, the time taken is always less when there is no wind, than when there is any constant wind.

4133 (*A. L. Putnam*). Let a, b, and c be integers with $b \neq 0$, and let d and f be the respective greatest common divisors of a and b and of c and b. Then if

$$a \not\equiv \pm d \pmod{b} \quad \text{and} \quad c \not\equiv \pm f \pmod{b},$$

there is an infinite number of integers k for which the equation

$$ax + bxy + cy = k$$

has no solution in integers.

4136 (*H. S. M. Coxeter*). The equation

$$\cos \pi x + \cos \pi y + \cos \pi z = 0, \qquad 0 \leqq x \leqq \tfrac{1}{2} \leqq y \leqq z \leqq 1,$$

has the trivial solutions $y = \frac{1}{2}$, $z = 1 - x$, and $y = \frac{2}{3} - x$, $z = \frac{2}{3} + x$. It has also the non-trivial solution $x = \frac{1}{5}$, $y = \frac{3}{5}$, $z = \frac{2}{5}$. Prove that it has no other rational solution.

4137 (*P. Erdös*). Given an integer $x \leqq n^2/4$ which has no prime factor greater than n, show that $n! \equiv 0 \bmod x$.

4138* (*G. Pólya*). Given two positive integers p and q define

$$S_{p,q} = \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2p} - \frac{1}{3} - \frac{1}{5} - \cdots - \frac{1}{2q+1} + \frac{1}{2p+2} + \cdots + \frac{1}{4p} - \frac{1}{2q+3} - \cdots - \frac{1}{4q+1} + \cdots,$$

$$P_{p,q} = \left(1 + \frac{1}{2}\right)\left(1 + \frac{1}{4}\right) \cdots \left(1 + \frac{1}{2p}\right)\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{5}\right) \cdots \times \left(1 - \frac{1}{2q+1}\right)\left(1 + \frac{1}{2p+2}\right) \cdots.$$

We obtain the series in which blocks of p positive terms alternate with blocks of q negative terms by rearranging the terms of the well known series

$$\frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \cdots = S_{1,1} = 1 - \log 2,$$

and the product by rearranging correspondingly

$$\left(1 + \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 + \frac{1}{4}\right)\left(1 - \frac{1}{5}\right) \cdots = P_{1,1}.$$

Show directly that $P_{p,q} = (p/q)^{1/2}$ and hence derive the well known result

$$S_{p,q} - S_{1,1} = \log (p/q)^{1/2}.$$

4142 (*G. Pólya*). Find a sequence of real numbers $a_1, a_2, \cdots$ so that $\sum_1^\infty a_n$ converges, $\sum_1^\infty a_n^3$ diverges, $\sum_1^\infty a_n^5$ converges $\cdots$. More generally, let C be an arbitrarily given (finite or infinite) class of positive integers. There exists a sequence of real numbers $a_1, a_2, \cdots, a_n, \cdots$ adapted to C so that, for $l = 1, 2, \cdots$, the series

$$a_1^{2l-1} + a_2^{2l-1} + \cdots + a_n^{2l-1} + \cdots$$

converges or diverges according as l does or does not belong to C.

4143 (*Paul Erdös*). Let $p_1 < p_2 < \cdots < p_n < \cdots$ be the consecutive primes. Prove that

$$\frac{p_n!}{p_n(p_n+1)\cdots(p_{n+1}-1)}$$

is always an integer except when $p_n = 3$.

4152* (*W. J. Taylor*). Prove the following trigonometric expansion for the binomial coefficient

$$\frac{N!}{\left(\frac{N+x}{2}\right)!\left(\frac{N-x}{2}\right)!} = \frac{2^N}{N}\sum_{m=1}^{N}\left(\cos\frac{m\pi}{N}\right)^N \cos\frac{m\pi x}{N}, \qquad -N < x < N.$$

4161* (*R. E. Gaines*). Along a straight road a miles long are n persons. What is the probability that no two persons are less than b miles apart?

4171 (*Tibor Rado*). Let $\mathbf{x}_n$, $n = 0, 1, 2, \cdots$ be a sequence of vectors in euclidean three-space such that $|\mathbf{x}_n| > \delta$ for all n, where δ is a fixed positive constant, and absolute value signs designate the length of the vector involved. Prove that the relation

$$|\mathbf{x}_n| + |\mathbf{x}_0| - |\mathbf{x}_n + \mathbf{x}_0| \to 0 \qquad \text{as } n \to \infty$$

holds if and only if there exists a sequence of positive scalars a_n such that $|a_n\mathbf{x}_n - \mathbf{x}_0| \to 0$ as $n \to \infty$.

4172 (*R. P. Agnew*). Prove or disprove the following statement involving cosine series. If $a_1, a_2, a_3, \cdots$ is a sequence of real constants, and if L_n is the length of the part of the graph of the function

$$f_n(x) = a_1 \cos x + a_2 \cos 2x + \cdots + a_n \cos nx$$

lying in the interval $-\pi \leqq x \leqq \pi$ then $L_1 \leqq L_2 \leqq L_3 \leqq \cdots$.

4173 (*Herbert Robbins*). Let $a < b$ be given numbers and let $f(t)$ be defined, continuous, non-negative, and strictly increasing for $a \leqq t \leqq b$. By the law of the

mean for integrals, for every $p>0$ there will exist a unique number x_p, $a \leqq x_p \leqq b$, such that

$$f^p(x_p) = \frac{1}{b-a}\int_a^b f^p(t)dt.$$

Find $\lim_{p\to\infty} x_p$.

4174 (*Irving Kaplansky*). Stone has called a ring "Boolean" if all its elements satisfy the equation $x^2=x$. Show that a ring in which $x^2=\pm x$ is either Boolean or the direct sum of a Boolean ring and the Galois field of three elements.

4176 (*H. S. M. Coxeter*). Prove the following two theorems in affine geometry of three dimensions:

(a) If all the faces of a convex polyhedron are parallelograms, their number is the product of two consecutive integers.

(b) If each face of a convex polyhedron has a center of symmetry, the whole polyhedron has a center of symmetry.

4185 (*B. M. Stewart*). In an unexplored region known as Wild Basin, a hunter found himself lost. But he had on hand a compass, and there were visible on two distant peaks fire ranger stations, A and B, whose bearings from his own cabin, O, he knew. From one observation point, C, the hunter took bearings on A and B; walking to another observation point, D, nearby, he took bearings on A, B, and C. Somehow he felt these seven bearings ought to enable him to find the direction homeward, that is, the bearing from D to O.

Show that the hunter's problem may be solved if he has either (1) mathematical tables or (2) a straightedge, in this case using the compass card as a protractor.

4186 (*Fritz Herzog*). It is known from the theory of probability that for x fixed, $0<x<1$, the largest of the $n+1$ terms ${}_nC_r x^r(1-x)^{n-r}$, $r=0, 1, \cdots, n$, is asymptotically equal to $[2\pi nx(1-x)]^{-1/2}$, as $n\to\infty$. Show that for all integral values of n and r with $n\geqq 1$, $0\leqq r\leqq n$ and for all values of x with $0<x<1$

$${}_nC_r x^r(1-x)^{n-r} < 1/[2enx(1-x)]^{1/2},$$

and that this represents the best inequality in the sense that the numerator on its right cannot be replaced by any number less than unity.

4190 (*Norman Anning*). If a, b, c, R are integers such that $a^2+b^2+c^2=R^2$, solve in integers the simultaneous equations

$$x^2 + y^2 + z^2 = R^2$$
$$ax + by + cz = 0.$$

4198 (*C. D. Olds*). Prove that

$$\frac{1}{(n-1)!}\int_n^\infty w(t)e^{-t}dt < \left(\frac{2}{e}\right)^n,$$

where t is real, n is a positive integer, and $w(t)=(t-1)(t-2)\cdots(t-n+1)$.

4202 (*Vladimir Karapetoff*). In a certain game of chance, consecutive numbers from 1 to n are written on a table. The same number of discs are provided with consecutive numbers written on them. The discs are turned with the numbers down so that the players cannot see the numbers written on them. A player covers all the numbers on the table with the discs at random, because he does not see the numbers on them. The discs are then turned over and the score is made on the basis of the number of discs whose numbers agree with the numbers on the table which they are covering. It is required to deduce an expression for the chance that k of the n discs covered the correct numbers.

4203 (*N. J. Fine*). If one is allowed n weighings on a beam balance, what is the maximum number A_n of coins, exactly one of which is bad, from which one can isolate the bad coin and determine whether it is heavy or light?

4212 (*H. F. Sandham*). Evaluate

$$\int_0^\infty \frac{e^{-x^2}dx}{(x^2+1/2)^2}.$$

4216 (*Herbert Robbins*). Write $\{x_n\}\tilde{c}\alpha$ if $\lim_{n\to\infty}(x_1+\cdots+x_n)/n=\alpha$. A function $f(x)$ is said to be C.c. (Cesaro continuous) at $x=\alpha$ if $\{x_n\}\tilde{c}\alpha$ implies $f(x_n)\tilde{c}f(\alpha)$. Show that if $f(x)$ is of the form $Ax+B$ then it is C.c. at every value of x, and that if $f(x)$ is C.c. at even a single value $x=\alpha$, then $f(x)$ is of the form $Ax+B$.

4220 (*Paul Erdös*). Let m be a positive integer with no prime factors greater than n and $m\leqq n^{(k+1)/2}$. Then m can be written as the product of k integers $\leqq n$; the exponent $(k+1)/2$ is the best possible.

4222 (*J. H. Butchart*). If points are numbered and if $\overline{12}$ denotes the distance from 1 to 2, then

$$\begin{vmatrix} 0 & \overline{12}^2 & 1 \\ \overline{21}^2 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = 2L^2,$$

where L is the length of the segment. Generalize this for the triangle and tetrahedron showing that the corresponding determinants are $-16A^2$ and $288V^2$ respectively, where A is the area of the triangle and V is the volume of the tetrahedron.

4229 (*Paul Erdös*). Let $f(z)=z^n+\cdots+a_n$, $g(z)=z^m+\cdots+b_m$ be two polynomials. Denote by A the region where $|f(z)|\leqq 1$ and by B the region where $|g(z)|\leqq 1$. Prove that A cannot properly contain B.

4235 (*Irving Kaplansky and D. C. Lewis*). Show that the determinant

$$\begin{vmatrix} (x-1)/1 & (x^2-1)/2 & \cdots & (x^n-1)/n \\ (x^2-1)/2 & (x^3-1)/3 & \cdots & (x^{n+1}-1)/(n+1) \\ \cdots & \cdots & \cdots & \cdots \\ (x^n-1)/n & (x^{n+1}-1)/(n+1) & \cdots & (x^{2n-1}-1)/(2n-1) \end{vmatrix}$$

is a constant times $(x-1)^{n^2}$.

4236 (*H. D. Grossman*). Write down two 1's, then a 2 between them, then a 3 between any two numbers whose sum is 3, then a 4 between any two numbers whose sum is 4, and so forth. Prove that the number of n's written down is $\phi(n)$, $(n>1$, where ϕ is Euler's totient).

4239 (*H. F. Sandham*). $AXBZ$ is a jointed rhombus connected with a fixed point O by two equal rods OA, OB. $OCZD$ is a jointed rhombus and YC, YD are equal rods. (Two Peaucellier cells, as it were, "cross joined.") Prove that as Y describes a circle, X describes a conic.

4240 (*Victor Thébault*). Determine the relations which must connect N, B, B', in order that the number N may be written with the same three digits in the system of numeration of base B as in the system of base B'. Having given B, find B' and N. Apply the results when $B=10$.

4249 (*W. B. Campbell*). A body is projected from a point O in a plane making an angle A with the horizontal, the direction of projection being in a vertical plane containing a line of greatest slope of the plane, and making an angle B with the upward direction of that line. If the plane be smooth and the body perfectly elastic, derive expressions for t_n, the time consumed in the nth flight, and for x_n, the coordinate of the point of impact at the end of the nth flight. Will it ever strike O again, and will any of its flights be vertical? What is the maximum x_n?

4250 (*Richard Bellman*). If

$$s_n=\sum_{k=1}^{n} a_k, \qquad \sigma_n=\sum_{k=1}^{n}\left(1-\frac{k}{n+1}\right)a_k,$$

and

$$\sum_{n=1}^{\infty} |s_n-\sigma_n|^k<\infty,$$

for any $k>0$, prove that $\sum_{n=1}^{\infty} a_n$ is convergent.

4252 (*Paul Erdös*). It is well known that $(2n)!/n!(n+1)!$ is always an integer. Prove that for every k there are infinitely many n's such that $(2n)!/n!(n+k)!$ is an integer.

4254 (*Paul Erdös*). We have seven points in the plane. Prove that we can always select three which do not form an isosceles triangle. For six points this does not necessarily hold. (If A, B, C are on a line we can define that they do not form an isosceles triangle if $AB \neq BC$.)

4255 (*G. Pólya*). A sequence $\{x_n\}$ is defined recursively, in terms of two numbers x_0 and x_1, by the formula

$$x_n = \frac{(n-1)g}{1+(n-1)g}\,x_{n-1} + \frac{1}{1+(n-1)g}\,x_{n-2},$$

where g is a given positive quantity. Find an expression for the limit of x_n as $n \to \infty$.

4258 (*H. F. Sandham*). Prove that the necessary and sufficient condition that four non-collinear points are such that each is the orthocenter of the other three, is

$$\pm\, 34\cdot 42\cdot 23 \pm 41\cdot 13\cdot 34 \pm 12\cdot 24\cdot 41 \pm 23\cdot 31\cdot 12 = 0,$$

where rs denotes the distance between the rth and sth points, and three of the signs differ from the fourth.

4259 (*Richard Bellman*). If

$$\sum_{k=1}^{\infty} \frac{n_k x^{n_k}}{1+x^{n_k}} = x\prod_{k=1}^{\infty}(1+x^{n_k}), \qquad |x| < 1,$$

show that, except perhaps for order, $n_k = 2^k$.

4263 (*Howard Eves and Paul Halmos*). Criticize the following alleged proof of the continuum hypothesis.

Let X be the set of all infinite sequences of 0's and 1's, and let E be an arbitrary uncountable subset of X. Corresponding to any finite sequence, $\{a_1, \cdots, a_k\}$, of 0's and 1's, write $E(a_1, \cdots, a_k)$ for the set of all sequences $\{x_n\}$ which belong to E and begin with $\{a_1, \cdots, a_k\}$. Since $E = E(0) + E(1)$, at least one of the two sets $E(0)$ and $E(1)$ is uncountable; write $a_1 = 0$ or 1 according as $E(0)$ is or is not uncountable. Then, in either case, $E(a_1)$ is uncountable. If a_i has already been defined for $i = 1, \cdots, k$, so that $E(a_1, \cdots, a_k)$ is uncountable, then write $a_{k+1} = 0$ or 1 according as $E(a_1, \cdots, a_k, 0)$ is or is not uncountable. The resulting infinite sequence $\{a_1, a_2, a_3, \cdots\}$ has the property that for any value of k it is true that $E(a_1, \cdots, a_k)$ is uncountable. Write E^* for the union of all $E(a_1, \cdots, a_k)$ for $k = 1, 2, 3, \cdots$; then E^* is a subset (in fact an uncountable subset) of E.

For certain positive integers k it is true that both $E(a_1, \cdots, a_k, 0)$ and $E(a_1, \cdots, a_k, 1)$ are uncountable: in fact this must happen for an infinite number of k's. (Otherwise, for a sufficiently large k, $E(a_1, \cdots, a_k)$ would not be uncountable, contrary to its construction.) Let $k_1, k_2, k_3, \cdots$ be the integers for which this is true, and write, for any $\{x_1, x_2, x_3, \cdots\}$ in E^*, $y_n = x_{k_n+1}$; then $\{y_1, y_2, \cdots\}$ is an infinite sequence of 0's and 1's. From the way in which the k_n are defined it follows that every possible sequence of 0's and 1's occurs as a y sequence, and that consequently the sequences $\{x_1, x_2, \cdots\}$ in E^* correspond (in possibly a many to one manner) to a set (*viz.* the set of all y sequences) having the power of the continuum. It follows that the cardinal number of E^* (and hence of E) cannot be less, and since E is a subset of X it cannot be greater. In other words it has been proved that every uncountable subset of a set having the power of the continuum has also the power of the continuum.

4264 (*G. Pólya*). Given $a>0$, $b>0$, and given that $f(x)$ is a non-linear function such that $f(0)=0$, $f(a)=b$, and that

$$f(x) \geqq 0, \qquad f''(x) \geqq 0, \qquad 0 \leqq x \leqq a,$$

give an analytic proof that

$$2\pi \int_0^a f(x)\left[1 + (f'(x))^2\right]^{1/2} dx < \pi b(a^2 + b^2)^{1/2}.$$

(The inequality becomes intuitive when both sides are interpreted as areas of curved surfaces.)

4267 (*C. F. Pinzka*). Let p be a prime greater than 3, and let r/ps be the sum of the harmonic series, $1+1/2+1/3+\cdots$, to p terms. Prove that p^2 divides $r-s$.

4268 (*Paul Erdös*). Let $a_1<a_2<\cdots$ be an infinite sequence of integers of upper density greater than $1/k$. (Denote by $f(n)$ the number of a's up to n, then the upper density is defined as $\overline{\lim}\, f(n)/n$ as $n\to\infty$.) Then for suitable t the equation

$$a_t = a_{i_1} + a_{i_2} + \cdots + a_{i_r} \qquad 1 < r < k$$

is solvable. In fact, there are infinitely many t with this property.

4270 (*S. H. Gould*). Let b be a fixed positive integer, $m=1, 2, \cdots$, and q irrational with $0<q<1$. Call the interval $(m, m+1)$ a gap if it does not contain a multiple of $b+q$. Prove that every set of b successive gaps contains exactly one multiple of $1+b/q$.

4273 (*I. S. Cohen*). Prove that for any positive odd integer n, $\cos\theta$ and $\sin\theta$ are rational functions of $\cos^n\theta$ and $\sin^n\theta$ with rational coefficients. Find the explicit expressions in the case $n=3$.

4278 (*Peter Ungar*). Construct two divergent series, $\sum a_k$ and $\sum b_k$ with $a_1 \geqq a_2 \geqq \cdots \geqq 0$, $b_1 \geqq b_2 \geqq \cdots \geqq 0$, but such that if $c_k \leqq \min(a_k, b_k)$, $c_k > 0$, then $\sum c_k$ is convergent.

4281 (*M. S. Knebelman*). Given an integer n. Show that an integer can always be found which contains only the digits 0 and 1 (in the decimal scale) and which is divisible by n. Is there an algorithm for finding the smallest such number?

4283 (*E. P. Starke*). The conjugate $\bar{z}$ of z, considered as a function of z, is nowhere analytic. Nevertheless, if C is an arbitrary circle or line, there exists a function $f(z)$ such that at every finite point of C, $f(z)$ is analytic and equal to $\bar{z}$. Consider also other curves for which a function exists having the same property.

4286 (*H. F. Sandham*). Prove that

$$\int_0^\infty \frac{\cos x^2 - \cos x}{x}\, dx = \tfrac{1}{2}\gamma,$$

where γ is Euler's constant.

4287 (*C. R. Phelps*). Show that for any given integer $k > 1$, there are an infinite number of perfect kth powers which cannot be written as the sum of a prime and a kth power. (This disproves a conjecture of Hardy and Wright, *Introduction to the Theory of Numbers*, p. 19.)

4288 (*J. W. Campbell*). Eddington referred to the following problem: If A, B, C, D each speak the truth once in three times (independently), and A affirms that B denies that C declares that D is a liar, what is the probability that D was telling the truth? He used the exclusion method of solution and arrived at the numerical result 25/71.

Prove that the correct probability is 13/41, and that this is also the probability that each of A, B, and C told the truth.

4290 (*P. T. Bateman*). The function $-\log\left|2 \sin \tfrac{1}{2}x\right|$ has the Fourier series

$$\cos x + \tfrac{1}{2}\cos 2x + \tfrac{1}{3}\cos 3x + \cdots.$$

Prove that no partial sum of the series is ever less than -1.

4293 (*H. F. Sandham*). Evaluate

$$\sum_{n=1}^{\infty} \frac{\left(\dfrac{3-\sqrt{5}}{2}\right)^n}{n^2}.$$

4295 (*Irving Kaplansky*). Show that any group with more than two elements admits an automorphism other than the identity.

4296 (*H. S. Wall*). It is known that when the continued fraction

$$x = p - \cfrac{q}{p - \cfrac{q}{p - \cdot}}$$

converges, then its value is the numerically larger root of the equation $x^2 - px + q = 0$. On the other hand, Newton's formula for computing the roots by successive approximation is

$$x_{k+1} = x_k - \frac{x_k^2 - px_k + q}{2x_k - p} = \frac{x_k^2 - q}{2x_k - p}, \qquad k = 0, 1, 2, \cdots.$$

Show that if x_0 is an approximant of the continued fraction, then $x_1, x_2, x_3, \cdots$ are approximants of the continued fraction.

4299 (*R. C. Lyness*). If the difference between two consecutive cubes is a square, then it is the square of the sum of two consecutive squares.

4300 (*Leo Moser*). Let $a_1, a_2, \cdots, a_n$ be n, not necessarily distinct, elements of a group of order n. Show that there exist integers p and q, $1 \leqq p \leqq q \leqq n$, such that

$$\prod_{i=p}^{q} a_i = 1.$$

4302 (*Joseph Rosenbaum*). Prove that if x and y have no common factor then every odd factor of

$$x^{2^n} + y^{2^n},$$

where n is a positive integer, is of the form $2^{n+1}m + 1$.

4303 (*G. T. Williams*). If

$$\theta_n = \int_0^1 x(x-1)\cdots(x-n+1)dx, \qquad \phi_n = \int_0^1 x(x+1)\cdots(x+n-1)dx,$$

show that $(-1)^n\theta_n = \phi_n - n\phi_{n-1}$ and find ϕ_n in terms of θ_n.

4305 (*H. F. Sandham*). Prove that

$$1 + \left(\frac{1+\frac{1}{2}}{2}\right)^2 + \left(\frac{1+\frac{1}{2}+\frac{1}{3}}{3}\right)^2 + \left(\frac{1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}}{4}\right)^2 + \cdots = \frac{17\pi^4}{360}.$$

4310 (*Paul Erdös*). Let $f(z) = z^n + \cdots$ be a polynomial of degree n. Denote by A_f the closed region (not necessarily connected) where $|f(z)| \leqq 1$. Prove that

there always exists a z_0 in A_f with $|f'(z_0)| \geqq n$. Equality occurs only for z^n.

4311 (*V. L. Klee, Jr.*). If k and x are positive integers, let $f_k^1(x) = k\phi(x)$, where $\phi(x)$ is Euler's totient. For $j = 2, 3, \cdots$, let $f_k^j(x) = f_k^{-1}[f_k^1(x)]$. Show that for $k \leqq 3$, the sequence $f_k^1(x), f_k^2(x), \cdots$, is eventually constant, while for $k \geqq 4$, the sequence is eventually monotonically increasing.

4312 (*R. C. Lyness*). In order to help our school savings campaign, Morris organized a lottery. Certain members of the savings group were each issued a book containing a gross of tickets. The tickets were sold for a penny each and the prize for the winning ticket was a number of sixpenny savings stamps. When I asked Morris how many stamps the winner got, he said that I could work it out for myself. All he need tell me was that before the winning ticket was drawn all the ticket sellers had assembled around a circular table and each had handed in his takings and his unsold tickets; that (a) no two had sold the same number; that (b) the number of unsold tickets returned by each seller was in every case equal to the product of the values in shillings of the tickets sold by his two neighbors at the table; and that (c) when the ticket sellers were rewarded with a penny each for their trouble the sum left over exactly bought the winner's stamps.

How many sellers were there and how many tickets did each sell?

4314 (*N. J. Fine*). Prove the identity:

$$1 + \frac{x}{(1-x)^2} + \frac{x^2}{(1-x)^2(1-x^2)^2} + \frac{x^3}{(1-x)^2(1-x^2)^2(1-x^3)^2} + \cdots = \frac{1 - x + x^3 - x^6 + x^{10} - \cdots}{[(1-x)(1-x^2)(1-x^3)\cdots]^2}.$$

4315 (*Albert Wilansky*). Consider the Clairaut differential equation $y = px + f(p)$, where $p = y'$ and f has a derivative. Prove that if f' is monotone the singular solution has exactly one point in common with any particular solution.

4317 (*Leo Moser*). Let G be an Abelian group and A a subset of order n, such that $a \in A$ implies $a^{-1} \in G - A$. Consider the n^2 (not necessarily distinct) elements of G of the form $a_i a_j$, a_i and a_j elements of A. Show that of these n^2 elements at most $\binom{n}{2}$ are elements of A.

4318 (*H. F. Sandham*). Show that

$$\frac{1}{(\sinh \pi)^2} + \frac{1}{(2\sinh 2\pi)^2} + \frac{1}{(3\sinh 3\pi)^2} + \cdots = \frac{2}{3}\left(\frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \cdots\right) - \frac{11\pi^2}{180}.$$

4319 (*Paul Erdös*). If p is a prime greater than 3 and if $n = (2^{2p} - 1)/3$, show that $2^n - 2$ is divisible by n.

4321 (*Paul Erdös*). Let $n_1<n_2<\cdots<n_k<\cdots$ be a sequence of integers such that $\lim_{k\to\infty} n_k/n_1n_2\cdots n_{k-1}=\infty$. Show that $\sum_{i=1}^{\infty} 1/n_i$ is irrational.

4322 (*Pedro A. Piza*). If p is a prime, prove that

$$\binom{n}{p} \equiv [n/p] \pmod{p},$$

where $\binom{n}{p}$ and $[n/p]$ have their usual meanings.

Prove also that when $[n/p]$ contains p^s as a factor, then $\binom{n}{p}$ is divisible by p^s.

4325 (*Orrin Frink*). Show that on every simple closed plane curve there are four points which are the vertices of a square.

4328 (*Victor Thébault*). Given a triangle ABC whose altitudes are AA', BB', CC'. Prove that the Euler lines of the triangles $AB'C'$, $A'BC'$, $A'B'C$ are concurrent on the nine-point circle at a point P which is such that one of the distances PA', PB', PC' equals the sum of the other two.

4329 (*A. W. Goodman*). Let θ be an irrational number, $a=e^{i\theta\pi}$. Prove that $f(z)=\sum_{n=0}^{\infty} a^{n^2}z^n$ has the unit circle as a natural boundary.

4330 (*Paul Erdös*). Let $a_1<a_2<\cdots$ be an infinite sequence of integers. Prove that there exists either an infinite subsequence in which no integer divides another or an infinite subsequence where each integer is a multiple of the preceding one.

4332 (*Paul Erdös*). Let $a_1<a_2<\cdots$ be an infinite sequence of integers. Prove that from the sequence a_i+a_j, $i=1, 2, \cdots, j=1, 2, \cdots$, one can always select an infinite subsequence such that no element divides another.

4336 (*Orrin Frink*). Find the arc of fixed length $l>2a$, lying below the x-axis and joining the points $(-a, 0)$ and $(a, 0)$, which includes between itself and the x-axis an area of lowest possible center of gravity. This will be the form actually assumed by a weightless flexible arc supported at its ends if it is holding water.

4337 (*R. M. Redheffer*). If the numbers R_{nk} are defined by

$$\frac{1-z^2}{\sin \pi z}\prod_{k=2}^{n} \sin(\pi z/k) \equiv \sum_{k=0}^{\infty} R_{nk}z^k,$$

prove that $\underline{\lim}\, R_{nk}^{-1/k}$ is equal to the first prime exceeding n.

4341 (*D. H. Browne*). The sequence $\{a\}=3, 7, 47, 2207, 4870847, \cdots$, used for the determination of primality of the Mersenne numbers, is usually defined by $a_{n+1}=a_n^2-2$. Show that it may also be defined by $a_k=f_{2^k+1}/f_{2^k}$, where the f's are the Fibonacci numbers, 1, 1, 2, 3, 5, 8, $\cdots$.

4342. (*R. J. Walker*). A spherical planet whose density at any point P is a

function only of the distance of P from the center of the planet has the following property. If a straight frictionless tunnel is bored between two points on the planet's surface the time required for an object to slide from one of these points to the other is independent of the positions of the points. Prove that the planet has constant density.

4345 (*Irving Kaplansky*). An element x in a ring is said to be right quasi-regular if there exists an element y with $x+y+xy=0$. It is evident that in a division ring, every element except -1 is right quasi-regular. Prove the converse: if every element in a ring A is right quasi-regular, with exactly one exception, then A is a division ring.

4346 (*N. S. Mendelsohn*). Prove that

$$n - 1 = \sum_{r=1}^{\infty} \left[\frac{n + 2^{r-1} - 1}{2^r} \right],$$

for any positive integer n. The brackets denote, as usual, the greatest integer function.

4348 (*D. A. Darling*). This problem was brought from Poland by Professor H. Steinhaus. It appears that Professor Banach was accustomed to carrying a box of matches in each of two coat pockets. To light his pipe, he would take a match from either box at random. The boxes contained originally n matches each. Banach's question is: when first a box is opened and found empty, what is the expected number of matches left in the other box?

4349 (*H. F. Sandham*). Prove that

$$\frac{2}{1} \Big/ \frac{5}{4} \Big/ \frac{8}{7} \Big/ \frac{11}{10} \cdots = \sqrt{3}.$$

4351 (*Albert Wilansky*). Let $f(x, y)$ be continuous for all (x, y). On each circle with center at the origin f assumes a minimum at certain points. Is the set of all such points throughout the plane connected?

4352 (*Paul Erdös*). Denote by $f(n; a_1, a_2, \cdots, a_k)$ the number of positive integers $m \leqq n$ which are either divisors or multiples of one of the a's $(1 < a_i \leqq n)$. Prove that

$$f(n; a_1, a_2, \cdots, a_k) \leqq f(n; 2, 3, \cdots, p_k),$$

where $2, 3, \cdots, p_k$ are the first k primes.

4353 (*H. F. Sandham*). Prove that

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \left[\frac{\log n}{\log 2} \right] = \gamma,$$

where $[x]$ denotes the integral part of x, and γ is Euler's constant.

4358 (*Paul Erdös*). Let $a_1<a_2<\cdots$ be a sequence of integers with the property that there does not exist an infinite subsequence of the a's in which no one divides another. Prove that the products

$$a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_k^{\alpha_k}, \qquad 0 \leqq \alpha_i,$$

have the same property. (If the a's are chosen to be the first k primes, then we have a well known result due to Dickson.)

4360 (*Free Jamison*). Any one of a group of airplanes may be refueled from any other. Each has a fuel capacity sufficient for a flight one-fifth the distance around the earth. Assuming that all have the same constant ground speed and the same rate of fuel consumption, that the only landing place and the only available fuel supply are at the home base, and that refueling time is negligible, find the minimum number of planes necessary so that one plane may fly around the earth and all return home safely.

4361 (*Melvin Dresher*). If $S_1, S_2, \cdots, S_m$ are m line segments parallel to the y-axis such that through every set of $n+2$ of them the locus of an nth degree polynomial can be passed, then there exists some nth degree polynomial whose locus intersects all m segments.

4363 (*Paul Erdös*). Let $a_1<a_2<\cdots$ be an infinite sequence of positive upper density (*i.e.*, $\underline{\lim}\, a_k/k<\infty$). Then there exists an infinite subsequence such that no element divides another. In fact, there exists an infinite subsequence $a_{i_1}, a_{i_2}, \cdots$ such that $\sum 1/a_{i_k}=\infty$ and no a_{i_k} divides any other.

4365 (*Paul Erdös*). Let $a_1<a_2<a_3<\cdots<a_k\leqq n$ be such that the least common multiple of any two a's exceeds n. Prove that

$$\sum_{i=1}^{k} \frac{1}{a_i} < 2.$$

4366 (*Joseph Rosenbaum*). Determine the condition which two concentric spheres must satisfy in order that a tetrahedron can be simultaneously inscribed in one and circumscribed about the other. Give a construction for the tetrahedron.

4371 (*Albert Wilansky*). Define $\theta(a, h)$ as the largest number θ satisfying

$$\text{(i) } 0 < \theta < 1, \qquad \text{(ii) } f(a+h) = f(a) + hf'(a+\theta h),$$

where $f(x)=x^2 \sin (1/x)$ for $x\neq 0$, $f(0)=0$.

Now set $\lambda(h)=[h\cdot\theta(0, h)]^{-1}$. Prove that as h tends to zero, $\lambda(h)$ tends to infinity in a step function manner; specifically, given $\epsilon>0$, there is a number $H(\epsilon)$ such that for every h with $|h|<H$ there is an integer $n(h)$ such that $|\lambda(h)-(n+\frac{1}{2})\pi|<\epsilon$.

4372 (*Ky Fan*). For what real values of x does the sequence $f_n(x)=\sin 7^n\pi x$ converge and what is the limit?

4375 (*N. J. Fine*). Let $((x))=x-[x]-\frac{1}{2}$. Prove that the sums

$$\sum_{n=1}^{m}((2^n x+\tfrac{1}{2}))$$

are uniformly bounded.

4381 (*R. J. Walker*). The inverse-square law has the property that the attraction, at an external point, due to a sphere of uniform density is the same as if the sphere were concentrated at its center. Are there any other laws of attraction which have this property?

4382 (*E. P. Starke*). Let $f_1(x)$ be Riemann integrable in the interval $0\leqq x\leqq M$ and let

$$f_{n+1}(x)=\int_0^x f_n(x)dx, \qquad n=1, 2, \cdots.$$

Show that

$$\phi(x)=\sum_{n=1}^{\infty} f_n(x)$$

is defined and continuous in the interval except perhaps at discontinuities of $f_1(x)$, and find a simple expression for $\phi(x)$.

4385 (*Peter Ungar*). The three-digit sections of the sequence 1110001011 represent all three-digit numbers in the binary system exactly once each. For a given positive integer n an analogous sequence is obtained in the following manner: write down n 1's to begin with, and in each subsequent place write 0 unless the n-digit section thus completed occurs previously, in which case put 1. Show that the resulting sequence of 2^n+n-1 digits has the same property as the case $n=3$ cited at the outset.

4387 (*Paul Bateman*). If $\sigma_r(n)$ denotes the sum of the rth powers of the divisors of the positive integer n, prove that

$$\sigma_r(n)\sigma_r(m)=\sum_{d\mid(n,m)} d^r\sigma_r(nm/d^2),$$

where d runs through all common divisors of n and m.

4388 (*Paul Erdös and W. H. Fuchs*). Let

$$f(z)=\prod_{i=1}^{n}(z-z_i), \qquad |z_i|\leqq 1.$$

Consider the set $|f(z)|\leqq 1$. Prove that it consists of at most $n-1$ components.

4389 (*F. J. Dyson*). Given N numbers a_m satisfying the N equations

$$\sum_{m=1}^{N} \frac{a_m}{m+n} = \frac{4}{2n+1}, \qquad n = 1, 2, \cdots, N,$$

prove that

$$\sum_{m=1}^{N} \frac{a_m}{2m+1} = 1 - \frac{1}{(2N+1)^2}.$$

4391 (*Paul Bateman*). Given a fixed integer k and a complex-valued function $f(n)$ defined on the positive integers and such that $f(n_1) = f(n_2)$ for $n_1 \equiv n_2 \pmod{k}$, $|f(n)| \leqq 1$ for all n, $f(n) = 0$ for $(n, k) > 1$, and $\sum_{n=1}^{k} f(n) = 0$. Show that

$$\left| \sum_{n=1}^{\infty} \frac{f(n)}{n} \right| < \log k.$$

4392 (*Paul Erdös*). Let $f(z)$ be analytic for $|z| \leqq 1$. Let z_0 be the point ($|z_0| = 1$) where $|f(z)|$ assumes its maximum on the unit circle. Prove that $f'(z_0) \neq 0$.

4394 (*H. F. Sandham*). Evaluate

$$\int_0^{\infty} \frac{\log x}{e^x + 1}\, dx.$$

4399 (*Ky Fan*). Let $f(n)$ and $g(n)$ be two sequences of natural numbers defined by the following three conditions:

(1) $f(1) = 1$.

(2) $g(n) = na - 1 - f(n)$, a being an integer > 4.

(3) $f(n+1)$ is the least natural number distinct from the $2n$ numbers $f(1), f(2), \cdots, f(n); g(1), g(2), \cdots, g(n)$.

Prove that there exist constants α and β such that $f(n) = [\alpha n]$, $g(n) = [\beta n]$. The brackets denote the greatest integer function.

4400 (*C. D. Olds*). Every positive root of the equation $\tan x = x$ can be expressed as follows: $x = (p + \frac{1}{2})\pi - \theta$ where p is an arbitrary integer $\geqq 0$ and

$$\theta = C_0\xi + C_1\xi^3 + C_2\xi^5 + \cdots, \quad \xi = 1/(p + \tfrac{1}{2})\pi.$$

The coefficients $C_0, C_1, C_2, \cdots$ are positive rational numbers. Show that, for n large,

$$C_n = 12^{-1/6}\Gamma(1/3)\pi^{-2/3} \cdot (\pi/2)^{2n+1}(2n+1)^{-4/3}(1 + \omega_n),$$

where $\omega_n \to 0$ when $n \to \infty$, so that the above series remains convergent even for $p = 0$.

4402 (*Carl Cohen*). Let the series of functions $\mathrm{dl}^{(n)}(x)$ be defined as

(1) $$\mathrm{dl}^{(n)}(x) = \int_0^x \mathrm{dl}^{(n-1)}(t)\,dt/t,$$

(2) $$\mathrm{dl}^{(1)}(x) = -\int_0^x \log(1-t)\,dt/t.$$

Show that for $|x| < \pi/2$,

(i) $$\tan x = \sum_{n=1}^{\infty} (2/\pi)^{2n}\left[\mathrm{dl}^{(2n-1)}(1) - \mathrm{dl}^{(2n-1)}(-1)\right]x^{2n-1},$$

(ii) $$\sec x = 1 + i\sum_{n=1}^{\infty} (2/\pi)^{2n+1}\left[\mathrm{dl}^{(2n)}(-i) - \mathrm{dl}^{(2n)}(i)\right]x^{2n}.$$

4413 (*Paul Erdös*). Let $a_1 = 2\cdot 3$, $a_2 = 3\cdot 5$, $a_3 = 5\cdot 7, \cdots, a_k = p_k \cdot p_{k+1}, \cdots$, where p_k is the kth prime. Denote by $f(x)$ the number of integers $\leqq n$ composed entirely of the a's (*i.e.*, the integers of the form $\prod a_i^{\alpha_i}$, $0 \leqq \alpha_i$). Prove that

$$f(n) = cn^{1/2} + o(n^{1/2}),$$

where $\frac{1}{2} < c < 1$.

4415 (*R. P. Boas, Jr. and W. K. Hayman*). Find all the values of α and β for which the series $\sum_{n=1}^{\infty} n^{\alpha} \sin n^{\beta}$ converges.

4416 (*D. J. Newman*). Let $a_1, a_2, \cdots$ be a sequence of integers such that $\sum_{n=1}^{\infty} 1/a_n$ diverges. Show that almost all integers have a factor in common with some a_n.

4418 (*A. C. Aitken*). Evaluate the determinant

$$a_n = \begin{vmatrix} b_1 & -1 & 0 \cdots & 0 \\ b_2 & b_1 & -2 \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ b_{n-1} & b_{n-2} & b_{n-3} \cdots & 1-n \\ b_n & b_{n-1} & b_{n-2} \cdots & b_1 \end{vmatrix}, \quad \text{where} \quad b_n = \frac{n^n}{n!}.$$

E4 (*W. F. Cheney, Jr.*) What is the simplest way to cut a wooden block 1 ft.$\times$1 ft.$\times$2 ft. into pieces which may be reassembled into a cube?

E24 (*R. K. Morley*). There are just three proper fractions with denominators less than a hundred which may be reduced to lowest terms by illegitimately canceling a digit. One of these is

$$\frac{26}{65} = \frac{2\not{6}}{\not{6}5} = \frac{2}{5}.$$

Find the other two and confirm the statement that there are no others.

E36 (*B. H. Brown*). Show that the thirteenth of the month is more likely to be Friday than any one of the other days of the week.

E46 (*B. H. Brown*). show that

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\cdots+\frac{1}{N}$$

is never an integer for any N.

E83 (*Morgan Ward*). Show that in any arithmetic progression of positive integers with common difference less than two thousand, at most ten consecutive integers can be primes.

E91 (*Morgan Ward*). Let d be the greatest common divisor of the two positive integers, a and b, with $a=a'd$ and $b=b'd$. Now if n is any integer greater than unity, show that (n^a+1) and (n^b-1) can not have any common factor greater than 2 as long as b' is odd.

E124 (*B. W. Jones*). Show that the volume generated by revolving a cube of edge a about one of its space diagonals is $\pi a^3/\sqrt{3}$.

E182 (*D. H. Lehmer*). Show that the infinite product

$$\left(1-\frac{i}{3}\right)^4\left(1-\frac{i}{17}\right)^4\left(1-\frac{i}{99}\right)^4\left(1-\frac{i}{577}\right)^4\cdots,$$

in which the successive denominators satisfy the recurrence $D_n=6D_{n-1}-D_{n-2}$, is purely imaginary.

E190 (*Fred Discepoli*). If a and n are positive integers greater than the positive integer b, then if $a^n+b^n=c^n$, c can never be an integer.

E198 (*J. E. Trevor*). A multiplication of a three-place number by a two-place number has the form

$$\begin{array}{r} p\ p\ p \\ \quad p\ p \\ \hline p\ p\ p\ p \\ p\ p\ p\ p \\ \hline p\ p\ p\ p\ p \end{array}$$

The p's are all prime digits, different from unity. Determine their values and show that the solution is unique.

E230 (*Meyer Karlin*). Prove that the sum of the series:

$$S_n = C_0^n - C_1^{n-1} + C_2^{n-2} - C_3^{n-3} + \cdots$$

will be $+1$, 0 or -1 according as n is a positive integer of the forms $6m$ or $6m+1$, $6m-1$ or $6m+2$, $6m+3$ or $6m+4$, respectively.

E231 (*A. A. Bennett*). In a certain bank there were eleven distinct positions; namely, in decreasing rank, President, First Vice-President, Second Vice-President, Third Vice-President, Cashier, Teller, Assistant Teller, Bookkeeper, First Stenographer, Second Stenographer, and Janitor. These eleven positions are occupied by the following, here listed alphabetically, Mr. Adams, Mrs. Brown, Mr. Camp, Miss Dale, Mr. Evans, Mrs. Ford, Mr. Grant, Miss Hill, Mr. Jones, Mrs. Kane, Mr. Long. Concerning them the following facts only are known:

1. The Third Vice-President is the pampered grandson of the president, but is disliked by both Mrs. Brown and the Assistant Teller.

2. The Assistant Teller and the Second Stenographer shared equally in their father's estate.

3. The Second Vice-President and the Assistant Teller wear the same style of hats.

4. Mr. Grant told Miss Hill to send him a stenographer at once.

5. The President's nearest neighbors are Mrs. Kane, Mr. Grant and Mr. Long.

6. The First Vice-President and the Cashier live at the exclusive Bachelors' Club.

7. The Janitor has occupied the same garret room since boyhood.

8. Mr. Adams and the Second Stenographer are leaders in the social life of the younger unmarried set.

9. The Second Vice-President and the Bookkeeper were once engaged to be married to each other.

10. The fashionable Teller is son-in-law of the First Stenographer.

11. Mr. Jones regularly gives Mr. Evans his discarded clothing to wear, without the elderly Bookkeeper knowing about it.

Show how to match correctly the eleven names against the eleven positions occupied.

E275 (*J. A. Benner*). In a certain town it began snowing before noon and continued at a constant rate until dark. At noon a crew of men set out along the highway, clearing the snow from it as they went. They cleared two miles in the first two hours, but only one mile in the next two hours. If the crew clears equal volumes of snow in equal times, at what time did it begin to snow?

E276 (*W. F. Cheney, Jr.*) The inside dimensions of a rectangular box with a lid on it are three feet, four feet, and five feet. A post in the form of a right circular cylinder nine inches in diameter just fits diagonally in the box, touching all six inner faces. How long is the post? (Note that the axis of the post, if prolonged, would miss the corners of the box.)

E285 (*D. L. MacKay*). If in triangle ABC, $\sin^2 A+\sin^2 B+\sin^2 C=1$, prove that the circumcircle cuts the nine-point circle orthogonally.

E312 (*D. L. MacKay*). If the scalene triangle ABC has its external angle bisectors at B and C equal, show that $(s-a)/a$ is the geometric mean of $(s-b)/b$ and $(s-c)/c$.

E319 (*Joseph Rosenbaum*). An electric light bulb is connected to n switches in such a way that the light is lit only when each switch is closed. Each switch is controlled by a push button, successive depressions of which will alternately open and close that switch. The push buttons are not provided with the usual marks, "on" and "off".

It is required to find in what order the push buttons should be pressed so that the greatest possible number of pushes which may be required to turn on the light will be as small as possible. Generalize the solution to cover the case in which each switch closes only at every p-th push.

E328 (*C. A. Murray*). In the equation $y=x^3+px^2+qx$, show how to determine all pairs of integral values of p and q for which the equation $y=0$ will have distinct integral roots and the two bend points, integral coordinates.

E332 (*Joseph Rosenbaum*). Prove that $1^p+2^p-3^p+4^p-5^p-6^p+7^p+8^p- \cdots +(2^n-1)^p=0$ for every positive integer $p<n$, where the sign of a term of the form m^p is negative or positive according as m is or is not one of the 2^k integers following 2^{k+1}, where k is any integer.

E345 (*F. E. Wood*). Let $S=a+b+c$, and $T=ab+ac+bc$, where a, b and c are the sides of a triangle. Show that $3T \leqq S^2<4T$. What are the analogous inequalities for a tetrahedron?

E364 (*A. V. Richardson*). If $(1+x)^n/(1-x)^3=a_0+a_1x+a_2x^2+ \cdots$, show that

$$a_0 + a_1 + a_2 + \cdots + a_{n-1} = \frac{n}{3}(n+2)(n+7)\cdot 2^{n-4}.$$

E378 (*J. L. Brenner*). Find the number of integral values of B which make B^2+m a perfect square, for any given, fixed, integer m.

E380 (*W. F. Cheney, Jr.*). If the radius of a circle is any odd prime, p, there are just two different primitive Pythagorean triangles circumscriptible about that circle. Show that, for each such pair of triangles:

(A) their shortest sides differ by one;
(B) their hypotenuses exceed their corresponding longer legs by one and by two, respectively;
(C) the sum of their perimeters is six times a perfect square;
(D) as p increases without limit, the ratio of their least angles approaches the limit 2;
(E) as p increases without limit, the ratio of their areas approaches the limit 2; and finally,

(F) the smaller triangle can always be placed inside the larger, so as not to touch it.

E395 (*E. P. Starke*). In high school geometry texts and elsewhere one frequently meets the statement that the reason for the straightness of the crease in a folded piece of paper is that the intersection of two planes is a straight line. This is fallacious. What is the correct reason?

E399 (*Victor Thébault*). Prove that the product of the first n positive integers $(1 \cdot 2 \cdot \cdots \cdot n)$ is divisible by their sum $(1+2+\cdots+n)$ if and only if $n+1$ is not an odd prime.

E400 (*H. S. M. Coxeter*). Show how to dissect a regular hexagon by straight cuts into the smallest possible number of pieces which can be reassembled to form an equilateral triangle (of the same area).

E402 (*Irving Kaplansky*). If n, r, and a are positive integers, the congruence $n^2 \equiv n \pmod{10^a}$ obviously implies $n^r \equiv n \pmod{10^a}$. (When such a number n has only a digits, it is called an automorphic number.) For what values of r does $n^r \equiv n \pmod{10^a}$ imply $n^2 \equiv n \pmod{10^a}$?

E413 (*H. T. R. Aude*). The graph of a cubic function $y=x^3+ax^2+bx+c$ crosses the x-axis at three distinct points, two of which are A and B. The lines AP and BQ are drawn tangent to the humps of the curve, the points of contact being P and Q. Show that the ratio of the distance AB to the horizontal distance from P to Q is constant.

E419 (*Victor Thébault*). In what direction must a billiard ball be hit in order to return to its starting point after a given even number of rebounds? Neglect spin of the ball.

E432 (*C. W. Trigg*). If a and b are the radii of two spheres, tangent to each other and to a plane, show that the radius x of the largest sphere which can pass between them is given by the formula $x^{-1/2}=a^{-1/2}+b^{-1/2}$.

E433 (*A. A. Bennett*). Two parallel vertical walls, separated by a distance of d feet, have level ground between them. Two ladders of length a and b feet, respectively $(a>b)$, abut each against a foot of one of these walls and lean against the other wall, crossing each other at a height of c feet above the ground. Show that a solution in integers is given by

$$ka = (su + tv)(s - t)(u + v), \qquad kb = (sv + tu)(s - t)(u + v),$$
$$kc = (su - tv)(sv - tu), \qquad kd = 2(stuv)^{1/2}(s - t)(u + v),$$

where s, t, u, v are any positive integers subject to the three conditions $u>v$, $sv>tu$, and $stuv$ is a perfect square, k being the greatest common divisor of the four right-hand members. What is the simplest particular solution in which a, b, c, d are all odd?

E435 (*David Segal*). Show that the congruence

$$\binom{2p-1}{p-1} \equiv 1 \pmod{p^2}$$

is a necessary and sufficient condition for p to be an odd prime.

E436 (*E. H. Johnson*). A tall rectangular piece of furniture, of length a and width b, is moved down a hallway of width c, and goes through a door whose width d barely allows its passage into an adjacent room. If we neglect the thickness of the wall, it is easily seen by the comparison of similar triangles that $d=ab/c$. If the wall has a thickness h, find the value of d in terms of a, b, c, and h.

E438 (*J. S. Frame*). If p is any odd prime, show that the decimal expansion of the fraction $1/p$ will repeat in $(p-1)/2$ digits or some factor thereof if and only if $p \equiv \pm 3^k \pmod{40}$.

E444 (*Harry Goheen*). Prove that there is no prime p such that $p^n+1=2^m$ if $n>1$, and that there is no prime p such that $p^n-1=2^m$ if $n>2$.

E456 (*Leopold Infeld*). What is the smallest popular vote by which a President can be elected in the U. S. A. under the present electoral system? Assumptions: N is the total popular vote, the popular vote in each state is proportional to the electoral vote (which you will have to look up); there are just two candidates.

E458 (*J. L. Brenner*). Prove that in any power of the matrix

$$\begin{pmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix},$$

two elements in the main diagonal will be the same. Show that the same result holds for any matrix (a_{rs}) in which $a_{r1}=a_{2r}$, $a_{1r}=a_{r2}$, $(r>2)$, and $a_{11}=a_{22}$.

E460 (*Henry Scheffé*). Let $s(n)=1+\frac{1}{2}+\frac{1}{3}+\cdots+1/n$ be the sum of the first n terms of the harmonic series. A well-known expression for $s(n)$ which does not formally involve the sum of n terms is the integral

$$\int_0^1 \frac{u^n-1}{u-1}\,du.$$

It is desired to write $s(n)$ in the form $s(n)=f^{(n)}(0)$. Find an expression for $f(x)$ in terms of the integral of an elementary function.

E468 (*W. R. Ransom*). The Fibonacci numbers, defined by $f_1=f_2=1$, $f_{j+1}=f_{j-1}+f_j$, are known to yield a puzzle in which a square of side f_n is cut into four pieces which can apparently be rearranged to form a rectangle $f_{n-1}\times f_{n+1}$.

Show that the same four pieces can be rearranged to form a figure which appears to consist of two rectangles $f_{n-1} \times 2f_{n-2}$ connected by a rectangle $f_{n-4} \times f_{n-2}$, the error being again one unit of area.

E481 (*J. A. Todd*). Let

$$\begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \\ X_3 & Y_3 & Z_3 \end{pmatrix}$$

be two matrices of non-vanishing numbers, the elements of the second being the co-factors of the corresponding elements of the first. Prove that the relation

$$\begin{vmatrix} x_1^{-1} & y_1^{-1} & z_1^{-1} \\ x_2^{-1} & y_2^{-1} & z_2^{-1} \\ x_3^{-1} & y_3^{-1} & z_3^{-1} \end{vmatrix} = 0 \quad \text{implies} \quad \begin{vmatrix} X_1^{-1} & Y_1^{-1} & Z_1^{-1} \\ X_2^{-1} & Y_2^{-1} & Z_2^{-1} \\ X_3^{-1} & Y_3^{-1} & Z_3^{-1} \end{vmatrix} = 0.$$

E491 (*E. T. Frankel*). Prove that $\sqrt[\sqrt{-1}]{-1} = 23\frac{1}{7}$ approximately.

E544 (*E. P. Starke*). Show that it is possible to construct a tetrahedron such that the length of every edge, the area of every face, and the volume all are integers.

E546 (*W. E. Buker*). Show that $\frac{1}{5}x^5 + \frac{1}{3}x^3 + \frac{7}{15}x$ is an integer for every integral value of x.

E554 (*J. L. Woodbridge*). Show that n cuts can divide a cheese into as many as $(n+1)(n^2-n+6)/6$ pieces.

E555 (*Howard Eves*). Consider a rectangular parallelopiped of dimensions $a \times b \times c$, made up of abc unit cubes. Imagine the edges of all these cubes replaced by material wires, common edges sharing the same wire. Prove that the exterior surface of the resulting network is of genus $p = 2abc + bc + ca + ab$ (*i.e.*, that it is topologically equivalent to a sphere with p handles).

E560 (*S. H. Gould*). In the fifth book of his *Laws*, the philosopher Plato, discussing the distribution of land in a colony, seeks a number divisible by every integer from 1 through 10 and chooses 5040. Show in general that if m and n are positive integers with $n < p$, where p is the smallest prime greater than m, then $m!$ is divisible by n except when $m = 3$.

E564 (*Ivan Niven*). Let a, b, and n be any positive integers such that n divides $a^n - b^n$. Prove that n divides $(a^n - b^n)/(a - b)$.

E598 (*H. S. Wall*). Let $g_1, g_2, g_3, \cdots$ be any numbers such that $0 < g_p < 1$, $(1 - g_p)g_{p+1} > \frac{1}{4}$, $(p = 1, 2, 3, \cdots)$. Prove that as $p \to \infty$, $\lim g_p = \frac{1}{2}$.

E610 (*Howard Eves*). (a) Show that all closed curves of the same constant diameter, d, have the same perimeter, πd. (b) What is the least area that a closed curve of constant diameter d may have? (Such a "curve of constant width" touches two parallel lines, distant d apart, drawn in any direction.)

E624 (*D. H. Browne*). Show that the integer nearest $n!/e$ is a multiple of $n-1$.

E628 (*W. C. Rufus*). Find the smallest positive integer, one half of which is a square, one third of which is a cube, and one fifth of which is a fifth power.

E670 (*C. D. Olds*). Sum the series $\sum_{r=1}^{2n-1} (-1)^{r-1} r/\binom{2n}{r}$.

E680 (*Gordon Pall*). Prove that a real determinant of order 6, with elements numerically not exceeding unity, cannot have a value greater than 160.

E681 (*W. B. Campbell*). From *Mrs. Miniver*: "She saw every relationship as a pair of intersecting circles. The more they overlapped, it would seem at first glance, the better the relationship; but this is not so. Beyond a certain point, the law of diminishing returns sets in, and there are not enough private resources left on either side to enrich the life that is shared. Probably perfection is reached when the area of the two outer crescents, added together, is exactly equal to that of the leaf shaped piece in the middle. On paper there must be some neat mathematical formula for arriving at this; in life, none."

Discuss the possibility of a unique solution for circles of given radii.

E687 (*Victor Thébault*). A heavy ball is gently dropped into a vase full of water, in the shape of a segment of a paraboloid of revolution. The size of the vase is given; that of the ball is such as to cause the maximum displacement. Find the radius of the ball.

E706 (*D. H. Browne*). For what values of n does $(a+b)^n$ yield only odd coefficients?

E710 (*D. H. Lehmer*). Find the inverse of the symmetric matrix of the nth order: $A = \|a_{ij}\|$, in which $a_{ij} = i/j$ for $i \leqq j$.

E711 (*H. S. M. Coxeter*). Suppose that the vertices of a polyhedron represent places that we wish to visit, while the edges represent the only possible routes. Hamilton considered the problem of visiting all the places, without repetition, on a single journey. (See, *e.g.*, W. W. R. Ball, *Mathematical Recreations and Essays*, London, 1939, p. 262.) This is easily solved for the pentagonal dodecahedron. Prove that it cannot be done for the rhombic dodecahedron.

E712 (*Donald Eves*). A man has twelve coins, all of which appear exactly alike, but one of which is counterfeit and does not weigh the same as a genuine coin. He has at his disposal a delicate set of balances, but no weights. How can he detect the false coin, and whether light or heavy, in not more than three weighings?

E735 (*Paul Erdös*). Six points can be arranged in the plane so that all triangles formed by triples of these points are isosceles. Show that seven points in the plane cannot be so arranged. What is the least number of points in space which cannot be so arranged?

E736 (*Paul Erdös*). Let $a_1<a_2<\cdots<a_k\leqq n$, where $k>[(n+1)/2]$, be k positive integers. Then $a_i+a_j=a_r$ is solvable.

E737 (*V. L. Klee, Jr.*). Establish the divergence of the series $\sum_{n=1}^{\infty} n^{-a}\cos(b\log n)$ for all values of $a\leqq 1$, regardless of the value of b.

E740 (*Esther Szekeres*). Let there be given five points in the plane. Prove that we can select four of them which determine a convex quadrilateral.

E744 (*Paul Erdös*). Let $a_1<a_2<\cdots<a_n\leqq 2n$ be n positive integers such that the least common multiple of any two is greater than $2n$. Then $a_1>[2n/3]$.

E750 (*Paul Erdös*). Find the number of intersections of the diagonals of a convex polygon of n sides.

E751 (*Alan Wayne*). Find the digits represented by the letters in the following addition, if no two different letters represent the same digit:

$$\begin{array}{r} F\ O\ R\ T\ Y \\ \quad T\ E\ N \\ \quad T\ E\ N \\ \hline S\ I\ X\ T\ Y \end{array}$$

E753 (*L. M. Kelly*). How can one convince a class in elementary analytics that if the inside of a race track is a non-circular ellipse, and the track is of constant width, then the outside is not an ellipse?

E756 (*G. Pólya*). Show that

$$\begin{vmatrix} a-x & 1 & 0 & 0 & \cdots & 0 \\ \binom{a}{2} & a-x & 1 & 0 & \cdots & 0 \\ \binom{a}{3} & \binom{a}{2} & a-x & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ \binom{a}{n} & \binom{a}{n-1} & \binom{a}{n-2} & \binom{a}{n-3} & \cdots & a-x \end{vmatrix}$$

$$=\binom{a+n-1}{n}-\binom{2a+n-2}{n-1}x+\binom{3a+n-3}{n-2}x^2-\cdots+(-1)^n x^n.$$

E774 (*Norman Anning*). Consider points on the median of a triangle. Through the centroid no straight line can be drawn which will cut off one third of the area. Through a point four fifths of the distance from vertex to base, four such lines can be drawn. Find points on the median at which the number of possible lines changes.

E776 (*L. R. Ford*). "Are those your children that I hear playing in the garden?" asked the visitor.

"There are really four families of children," replied the host. "Mine is the largest, my brother's family is smaller, my sister's is smaller still, and my cousin's is the smallest of all. They are playing drop the handkerchief," he went on; "they prefer baseball but there are not enough children to make two teams. Curiously enough," he mused, "the product of the numbers in the four groups is my house number, which you saw when you came in."

"I am something of a mathematician," said the visitor, "let me see whether I can find the numbers of children in the various families." After figuring for a time he said, "I need more information. Does your cousin's family consist of a single child?" The host answered his question, whereupon the visitor said, "Knowing your house number and knowing the answer to my question, I can now deduce the exact number of children in each family."

How many children were there in each of the four families?

E780 (*G. Pólya*). A lampshade has the shape of a frustum of a right circular cone. Its perimeter is P at the bottom, p at the top, and its slant height is s. Show that such a lampshade can be cut out in one piece from a rectangular sheet of paper with dimensions P and $s+p(P-p)/8s$. You can even save paper for a flap to glue the ends together, except for the limiting case where $P=p$, when not a bit of paper is wasted.

E785 (*R. J. Walker*). Each of $n-1$ tanks, $T_1, \cdots, T_{n-1}$, holds V gallons of water, and an nth tank, T_n, hold V gallons of a salt solution containing M pounds of salt. Liquid is circulated at the rate of g gallons per minute from T_n to T_{n-1}, T_{n-1} to $T_{n-2}, \cdots, T_2$ to T_1, T_1 to T_n. How much salt is in T_n after t minutes?

E788 (*Leo Moser*). Consider a map on a spherical surface where the countries are determined by n great circles of which no three are concurrent. Show that if n is a multiple of four it is impossible to make a trip visiting each country once and only once, if travelling along a boundary or crossing at a boundary point of more than two countries is forbidden.

E791 (*G. W. Walker*). The court mathematician once received his salary for a year's service all at one time, and all in silver "dollars", which he proceeded to arrange in nine unequal piles, making a magic square. The king looked, and admired, but complained that there was not a single prime number in any

of the piles. "If I had but nine coins more," said the mathematician, "I could add one coin to each pile and make a magic square with every number prime." They investigated, and found that his was indeed true. The king was about to give him nine "dollars" more, when the court jester said, "Wait!". Then the jester subtracted one coin from each pile instead; and they found in this case also a magic square with every element a prime number. The jester kept the nine "dollars". How much salary must the mathematician have been receiving?

E793 (*Joseph Rosenbaum*). With straight edge alone construct a hexagon which can possess both an inscribed and a circumscribed conic.

E806 (*Leo Moser*). Lewis Carroll once proposed the following problem.

"Two travellers spent from 2 o'clock till 9 in walking along a level road, up a hill, and home again; their pace on the level being x miles per hour, up hill y, and down hill $2y$. Find the distance walked."

In the original problem x and y were given integers. Deduce a solution to the original problem without *a priori* knowledge of what these integers are.

E812 (*Monte Dernham*). Find the shortest perimeter common to two different primitive Pythagorean triangles.

E813 (*C. W. Trigg*). Let S be the sum of the integer elements of a magic square of order three, and let D be the value of the square considered as a determinant. Show that D/S is an integer.

E817 (*E. V. Hofler*). If the graph for a polynomial of the fourth degree has two real points of inflection, then the secant through these two points and the curve will bound three distinct areas. Show that two of these areas are equal and the largest area is equal to the sum of the other two.

E819 (*H. F. Sandham*). If $S_n=1/1+1/2+\cdots+1/n$, prove that $\gamma<S_p+S_q-S_{pq}\leqq 1$, where γ is Euler's constant.

E824 (*E. P. Starke*). We modify the harmonic series by taking the first term positive, the next two negative, the next three positive, *etc.* Show that this modified series is convergent.

E827 (*Leo Moser*). Show that the reciprocal of every integer greater than 1 is the sum of a finite number of consecutive terms of the infinite series $\sum_{j=1}^{\infty} 1/j(j+1)$.

E832 (*V. E. Dietrich*). If a circle has a center with at least one irrational coordinate, then there are at most two points on the circle with rational coordinates.

E834 (*Don Walter*). Show that

$$F_n = \begin{vmatrix} 1 & -1 & 1 & -1 & 1 & -1 \cdots \\ 1 & 1 & 0 & 1 & 0 & 1 \cdots \\ 0 & 1 & 1 & 0 & 1 & 0 \cdots \\ 0 & 0 & 1 & 1 & 0 & 1 \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \cdots \end{vmatrix},$$

where F_n is the nth term of the Fibonacci sequence, 1, 1, 2, 3, 5, $\cdots$, x, y, $x+y$, $\cdots$, and the determinant is of order $n-1$.

E846 (*H. J. Hamilton*). The following is typical of many characterizations of the principal part of an infinitesimal which are to be found in elementary calculus texts.

"If an infinitesimal consists of two or more terms of different orders, the term of lowest order is called the principal part of the infinitesimal."

Show that this is not definitive and give a valid definition.

E853 (*C. S. Ogilvy*). If $y_1=x$, $y_2=x^{y_1}$, $\cdots$, $y_n=x^{y_{n-1}}$, what is the maximum x for which $\lim_{n\to\infty} y_n$ exists, and what is this limit?

E854 (*Jerome C. R. Li*). Show that $\pi = \sum_{n=0}^{\infty} (n!)^2 2^{n+1}/(2n+1)!$.

E860 (*Leo Moser*). Show that if all the faces of a polyhedron have central symmetry then it can be dissected by a finite number of plane cuts and the pieces fitted together to form a solid cube.

E879 (*Josef Langr*). Let S_1, S_2, S_3 be the midpoints of three concurrent cevians of triangle ABC. Let S_2S_3, S_3S_1, S_1S_2 meet the sides BC, CA, AB in A_1, B_1, C_1; A_2, B_2, C_2; A_3, B_3, C_3 respectively. Show that (1) A_2, A_3; B_3, B_1; C_1, C_2 are isotomic points on the segments BC, CA, AB, (2) A_1, B_2, C_3 are collinear (3) $A_2, A_3, B_3, B_1, C_1, C_2$ lie on a conic.

E880 (*Peter Ungar*). Let n points be given in the plane, not all on a straight line. The shortest closed route connecting them is a simple polygon.

E888 (*H. D. Grossman*). Show how to cut a hole in a cube through which another cube of equal size can pass.

E919 (*Leo Moser*). Show that the greatest common divisor of a and b is given by

$$(a, b) = \sum_{m=0}^{a-1} \sum_{n=0}^{a-1} {}^{\dagger} (1/a) e^{2\pi i bmn/a}.$$

E920 (*T. G. Room*). If S and T are any two square matrices of the same order, and the necessary matrices are non-singular, then

$$(I+S)^{-1}(S+T)(I+ST)^{-1}(I+S) = (I-S)(I+TS)^{-1}(S+T)(I-S)^{-1}.$$

E924 (*D. J. Newman*). Find $\lim_{n\to\infty} n \sin(2\pi e n!)$.

E929 (*H. S. Shapiro*). Given three non-concurrent straight lines l_1, l_2, l_3 in the plane. Let T_i denote reflection in l_i and set $T = T_1T_2T_3$. Show that T^2 is a translation.

E930 (*Gordon Raisbeck*). If $\prod_{i=1}^{n}(x+r_i) \equiv \sum_{j=0}^{n} a_j x^{n-j}$, show that

$$\sum_{i=1}^{n} \tan^{-1} r_i = \tan^{-1} \frac{a_1 - a_3 + a_5 - \cdots}{a_0 - a_2 + a_4 - \cdots}$$

and

$$\sum_{i=1}^{n} \tanh^{-1} r_i = \tanh^{-1} \frac{a_1 + a_3 + a_5 + \cdots}{a_0 + a_2 + a_4 + \cdots}.$$

E931 (*H. D. Larsen*). Ten balls numbered from 0 to 9 inclusive are placed in an urn. Five of the balls are then drawn at random (without replacement) and arranged in a row. What is the probability that the number thus formed is divisible by 396?

E937 (*I. N. Herstein*). If p is a prime and $n \geqq p$, then

$$n! \sum_{p\nmid i+j=n} 1/p^i i!j! \equiv 0, \bmod p.$$

E943 (*S. H. Gould*). By analogy with the motion of the planets, it seems natural to assume for any central motion that, at least if the particle never passes through the center, (a) the distance of the particle from the center is a maximum only when the velocity of the particle is a minimum, (b) the velocity is a maximum only when the distance is a minimum. Prove (a) and give a counter-example of (b).

E945 (*Leo Moser*). If all the faces of a convex polyhedron have central symmetry show that there are at least eight vertices where exactly three edges meet. (The cube has exactly eight such vertices.)

A CLASSIFICATION OF MONTHLY PROBLEMS
(1918–1950)

Note. All problems proposed in the MONTHLY between the years 1918 and 1950, inclusive, appear in the following classification. The classification is first divided into the three major divisions of *algebra*, *geometry*, and *analysis*; each of these major divisions is then further divided into subdivisions corresponding roughly to areas of instruction, arranged in general order of advancement. Most of the subdivisions are further broken down into appropriate topics, alphabetically arranged within the subdivisions. Topics containing four or fewer proposals, and any residue of unclassified problems in a subdivision, appear under the topic heading *miscellaneous* near the end of the subdivision. The topic for a given problem was determined principally by the mathematical tools used in the published solution of the problem; many problems accordingly appear under more than one topic. The numbers of the four hundred best problems are printed in italics. At the end of each of the three major divisions is a list of the unsolved problems in that division. (These are the problems proposed between 1918 and 1950, inclusive, for which solutions are still lacking. There are other problems proposed in the same period that could also be classified as unsolved but the statements of these problems have been followed up by either a partial solution, further comment, or an Editorial Note.)

ALGEBRA

Elementary algebra

Indeterminate equations—2790, E57, E106, E115, E135, E158, E163, E249

Verbal problems—2858, 3379, E135, E147, E163, E211, E222, E234, E249, E273, E366, E368, E408, E581, E641, E671, E872

Miscellaneous (equations; maxima and minima; simultaneous equations; etc.)—2662, E39; *2871*, 4180; 3050, E234, E686; 3651

College Algebra

Factoring—2881, *3673*, 3864, E162, E343, E614, E690, E716

Identities—2800, *3418*, 3545, 4062, E26, E152, E355, E787

Inequalities—*3207*, *3703*, E19, E186, *E190*

Permutations and combinations—3561, *3731*, 3761, 4277, E28, E97, E213, E272, E306, E360, E579, E589, E599, E650, E678, E772, E777, E801, E907

Progressions—2760, 2766, 2896, 3626, E11, E15, E48, E82, *E83*, E132, E186, E317, E324, E576, E584, E655, E658, E696, E726, E745, E768, E876, E877, E926

Simultaneous equations—2797, 2880, 3360, E235, E594, *E832*

Miscellaneous (equations; rate problems; etc.)—2724, E616, E736, E842; 2974, 3218, E466; *3528*, *4312*, E61, E390, *E456*, *E546*

Mathematics of finance—2744, 2857, 3113, 3306, E158, E169

Higher algebra

Binomial coefficients—3081, 3095, 3377, *3399*, *3414*, 3426, 3454, 3465, *3468*, 3517, 3596, *3612*, *3625*, 3629, 3641, *3645*, *3674*, 3679, 3691, 3701, 3710, *3719*, 3748, 3756, 3803, 4120, 4130, *4152*, 4189, 4211, 4276, *4322*, 4356, 4378, 4380, E21, E101, E155, E205, *E230*, E300, E320, E335, E349, E360, E406, E416, E429, E445, E464, E484, E629, *E670*, *E706*, *E756*, E794, E799, E839, E864

Continued fractions—2695, 3143, 3206, *3430*, *3459*, 3467, 4078, 4247, *4296*, *E182*, E308, E603, E620, E621

Finite series—2664, 2822, 2983, 3063, 3220, 3377, 3426, 3513, *3536*, 3596, 3629, 3641, 3688, 3691, 3701, *3719*, 3756, 3781, *3792*, 3803, *4050*, *4051*, *4118*, 4120, 4130, *4152*, 4155, 4183, 4189, 4191, 4211, 4276, *4290*, 4291, 4356, *4365*, 4380, *4387*, 4390, 4396, E27, E101, E155, E165, E205, *E230*, E300, *E332*, E349, E406, E422, E429, E445, E514, E550, E629, E636, *E670*, E700, E746, E760, E799, *E819*, *E827*, E839, E864, *E919*

Polynomials—2722, *2908*, 3518, *3678*, 3702, 3711, 3844, 3940, 3950, *3980*, 4410, E290, E484, E869

Miscellaneous (factoring; inequalities; partial fractions; progressions; quadratic forms; simultaneous equations; etc.)—2842, *3673*, 3709; 2829, *3764*, 4155, *4270; 3536*, E274, E290; 2699, 2710, 2764, 4031; 3406, *3705;* 2666, 2754, *4389*, E412; 2878, 3020, *3242*, 3350, 3900, 3920, E765

Probability

Cards—2714, 2942, 2943, 3266, *3652*, E490, E727, E798

Dice—2783, 3210, 3958, *4093*, E771, E850, E925

Geometrical—2692, 2782, *3539*, 3948, E119, E626

Miscellaneous (balls; coins; etc.)—2698, 3762, 3799, *E931;* 3046, *3872*, E683, E796; 3107, 3164, 3383, 3615, *4108*, 4146, *4161*, 4177, *4186*, *4202*, *4288*, *4348*, 4377, 4412, *E36*, E54, E126, E347, E504, E531, E611, E656, E717, E719, E725, E754, E811

Determinants

Evaluation—2740, *2774*, 3081, *3118*, *3468*, 3516, *3552*, *3645*, *3667*, 3679, *3687*, 3710, *3747*, 3748, *3783*, *3784*, 3873, *3919*, 3994, *4023*, 4071, 4098, *4101*, 4125, *4222*, *4235*, 4362, 4367, 4383, *4418*, E280, E648, E674, *E680*, *E756*, E789, E794, *E834*, E857

Identities—2807, 3065, 3154, 3309, 3349, 3517, 4159, E625

Special determinants—2908, 2939, *3140*, 3601, *4009*, 4183, E910

Miscellaneous (differential equations; differentiation; factorization; etc.)—3039, 3340, 3562; *3249*, 3714; 3086, 3564, E65, *E813;* 2938, 3152, *3506*, *3514*, E104, *E481*, E644, E660, E775

Theory of equations

Approximation of roots—2803, 2836, 3515, *4296*, E616

Cubics—2828, 2852, 2996, *3092*, 3109, 3476, *E328*, E574

Roots and coefficients—2678, 2874, *3032*, *3034*, 3124, 3427, 3497, 3662, 3800, *4009*, E221, E351

Simultaneous equations—2833, 2835, 3239, 3259, *3506*
Solutions—3523, *3621*, 3730, 3894, *3907*, E223, E228
Miscellaneous (bounds on roots; quartics; roots of unity; etc.)—3162, E743, E884; 3036, *3173*, E588; 2839, 2864, 3672, E518; 3135, 3311, 3391, 3631, *4136*

Number theory
Arithmetico-geometrical—2661, 2671, 2738, 2741, 3386, 3733, *3776*, 3944, 3955, *4047*, 4205, E121, E283, E308, E322, E331, E352, E374, E379, E410, E418, E480, E515, E525, *E544*, E595, E620, E695, E747
Congruences—3125, 3198, 3208, 3403, *3445*, 3472, 3488, *3682*, *3707*, *3721*, *3771*, 3933, 4121, 4297, *4319*, *4322*, *E402*, E425, *E435*, E446, E462, E464, E654, E792, E896, E911, *E937*
Digit problems—2906, *4054*, 4055, 4100, 4103, 4107, 4114, 4123, 4135, 4140, 4144, 4157, 4164, 4184, 4217, 4227, 4230, 4237, *4240*, 4244, 4269, *4281*, 4327, 4355, *4385*, E13, E*24*, E42, E63, E69, E89, E94, E116, E118, E127, E136, E148, E159, E172, E176, E180, E187, E193, E195, E204, E207, E218, E232, E237, E240, E242, E246, E250, E261, E271, E278, E282, E284, E291, E299, E377, E389, E393, E404, E578, E582, E602, E608, E622, E638, E658, E718, E802, E856, E942
Diophantine equations—2700, 2729, 2751, *2784*, 2790, 2927, 3166, 3182, *3231*, *3251*, 3439, 3449, *3459*, 3467, 3584, 3677, *3692*, 3735, 3854, 3855, 3865, 3917, *4047*, 4123, *4190*, 4284, *4299*, 4401, E51, E55, E143, E151, E210, E227, E253, E313, E326, E353, E357, E371, E430, *E433*, E494, E509, E596, E640, E668, E682, E685, E695, E702, E755, *E806*, E816, E823, E901, E917, E926, E941, E942
Divisibility—3049, 3085, 3117, 3194, 3203, 3530, 3561, *3582*, 3638, *3687*, *3739*, 3761, *3885*, *3903*, *4083*, *4137*, *4143*, *4220*, 4226, *4252*, 4257, *4267*, *4281*, *4302*, *4330*, *4332*, 4339, *4352*, *4358*, *4363*, *4387*, *4416*, E28, *E46*, E77, *E83*, *E91*, E95, E123, E294, E369, *E399*, *E546*, *E560*, *E564*, *E628*, E655, E684, E692, E716, *E744*, E745, E768, E779, E808, E822, E903, *E919*, E932
Euler ϕ function—3058, 3122, *4002*, *4068*, 4121, 4221, *4236*, *4311*, 4327
Fibonacci numbers—2809, *4341*, *E468*, *E834*, E852
Partitions—*4067*, 4073, 4246, 4304, 4340, 4390, E883
Pythagorean numbers and triangles—2755, *3594*, 3901, *4047*, 4110, 4205, E18, E67, E73, E265, E283, E324, E327, E362, *E380*, E472, *E812*, E828
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E887, E916, E921

Special numbers—3827, 3915, 3926, *4012*, 4020, 4048, E225, E338, E414, E427, E431, E440, E452, E492, E502, E506, E532, E538, E548, E562, E848

Squares—2832, *4054*, 4055, 4144, 4157, 4184, 4227, 4230, 4237, 4244, 4284, 4355, E25, E42, E127, E154, E172, E176, E187, E189, E193, E195, E207, E218, E232, E237, E240, E242, E246, E253, E261, E267, E278, E291, E362, *E378*, E385, E393, E492, E534, E572, E582, E592, E822, E909

Miscellaneous (algebraic number theory; euclidean algorithm; Fermat's Last Theorem; Möbius inversion formula; Pell equation; powers; power residues; quadratic residues; repeating decimals; triangular numbers; etc.)—2914, 2916; 2844, E852; *3023*, *3305*, *E190*, E729; *4101*, *4104*; 4123, E603, E833; 4333, E373, E663; 4307, E575; *4337*, *E444*, E663; 3121, *E438*; *2930*, E584; 3480, E25, E330, E496; 2700, 2914, 2916, 3150, 3471, 3626, 3725, *3820*, 3825, *4399*, 4404, E153, E340, E375, E397, E451, E512, E676, E818, E890, E898, E921

Abstract algebra

Groups—3424, 3495, 3661, 3742, 3775, 4010, *4295*, *4300*, *4317*, E516

Matrices—*2884*, 3605, 3646, *3700*, *3720*, *3747*, 3796, *3824*, 3927, *4023*, *4042*, *4124*, 4126, 4168, *E458*, E579, E634, *E710*, E733, *E920*

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Some special topics

Cryptarithms and arithmetical restorations—3212, E1, E7, E10, E20, E22, E23, E30, E37, E43, E49, E58, E71, E78, E99, E105, E130, E140, E160, E164, E173, E184, E192, *E198*, E217, E251, E258, E265, E297, E317, E333, *E751*, E891

Enumeration problems—2681, *2688*, 3107, 3115, 3264, 3363, *3421*, 3432, 3454, *3792*, *3893*, *3918*, *3954*, 4148, *4203*, 4291, *4360*, E28, E112, E178, E185, E208, E272, E295, E306, *E319*, E448, E500, E505, E508, E516, E524, *E554*, E565, E579, E589, E599, E678, E716, E719, *E750*, E754, E772, E773, E777, E783, E801, E845, E907, E927

Games, recreational problems, puzzles; fallacies and paradoxes—3651, *3918*, E14, E81, E358, E631, E651, *E712*, E717, E773, E811, E816, E850, E866, E916; 2773, 2870, *4263*, E2

Logic and reasoning problems—2894, *3734*, *E231*, E601, E651, *E712*, *E776*, E821

Magic squares—E428, E496, E674, E733, E764, *E791*, *E813*

Unsolved algebra problems—2663, 2717, 2725, 2727, 2730, 2778, 2787, 2788, 2798, 2810, 2817, 2826, 2854, 2867, 2886, 2905, 2909, 2915, 2935, 2940, 2953, 2970, 2984, 2985, 3075, 3082, 3097, 3099, 3127, 3133, 3141, 3167*,† 3172*, 3183*, 3185*, 3196*, 3204*, 3209, 3271, 3294, 3304, 3312, 3329, 3347, 3354, 3358, 3364, 3376, 3423, 3450, 3457, 3484, 3494, 3498, 3542, 3549, 3553, 3555, 3606, 3632, 3655, 3663, 3685, 3708, 3717, 3779, *3834*, *3835*, 3870, 3876, 3880, 3881,

† For explanations of asterisk, see p. 79.

3910, 3951, 3997, 4003, 4022, 4097, 4139, 4207, 4261, 4289, 4326, 4350, E518, E534, E590, E604, E825

GEOMETRY

Elementary plane geometry

Circles—2679, 2814, 2823, 2900, 2901, 3021, 3074, 4253, E59, E107, E142, E171, E203, E409, E470, E571

Maxima and minima—3250, 3567, 3576, 3769, *E880*

Quadrilaterals—3064, 3080, 3359, 3791, E11, E38, E117, E129, E206, E334, E344, E346, E359, E391, E486, E643, *E740*, E767, E851, E870

Triangles—2660, 2735, 2736, 2765, 2834, 2993, 3326, 3341, *3726*, *3746*, *4254*, E52, E64, E68, E137, E139, E144, E150, E166, E179, E220, E281, E301, E305, E307, *E345*, E398, E497, E545, E626, E878

Miscellaneous (loci; numerical problems; regular polygons; similar triangles; etc.)—3479, E56; 3246, 3297; 3322, *4086*, E786; 3325, E214; E31, E455, *E735*

Elementary solid geometry

Polyhedra—E341, *E345*, *E711*, E859, *E888*

Miscellaneous (volumes; etc.)—3146, 3256; 2982, 3263, 3373, 3375, E79, E388, *E395*, *E432*, E689, E748

College plane geometry

Circles—3153, 3455, 3477, 3525, 3583, 3622, 3727, 3753, 3758, 3769, 3793, 3795, 3871, 3886, 3925, 4038, 4056, 4059, 4088, 4092, 4145, 4313, 4323, E93, E113, E238, E248, E262, E499, E526, E571, E591, E597, E705, E728, E886

Euler line—3228, 3956, 4057, *4328*, E102, E259, E547

Harmonic points, poles and polars—3213, 3338, 3436, 3751, 3828

Polygons—3867, 3886, 3970, 4008, 4025

Quadrilaterals and quadrangles—2723, 3295, 3371, 3609, 3738, 3788, 3890, 4194, 4323, E268, E420, E574, E619

Simson line—2776, 3461, *3658*, 4165, 4194, 4309, E535, E561

Triangles—2720, 2961, 3258, 3267, 3273, 3321, 3328, 3344, 3387, 3390, 3396, *3440*, 3508, 3525, 3628, 3634, 3640, 3653, 3675, 3751, 3753, 3758, 3772, *3780*, 3793, 3817, 3828, 3832, 3838, 3848, 3857, 3871, 3874, 3882, 3883, 3889, 3891, 3896, 3902, 3904, 3921, 3925, 3929, 3945, 3960, 3988, *3990*, *4019*, 4025, 4026, 4030, 4038, 4056, 4057, 4060, *4070*, 4074, 4076, 4089, 4092, 4102, 4119, 4134, 4145, 4169, 4179, 4182, 4196, 4210, 4260, 4274, 4294, 4301, 4344, 4386, 4401, E75, E122, E131, E236, E269, E350, E361, E367, E374, E383, E407, E411, E417, E459, E467, E469, E487, E521, E527, E537, E573, E613, E699, E720, E730, E742, E752, E803, E820, E849, *E879*, E905, E915

Miscellaneous (concurrency and collinearity; isogonal conjugates; circles of Apollonius; etc.)—2715, *4065*, E113, E831; 3797, 4038; 3234, 3973; 3937, *4325*, E323, E372, *E929*

College solid geometry

Cones—3581, 3989, 3993, E749, *E780*

Loci—2967, 3431, 3646, 3847, 3961, 4200, E33, E255, E426, E463, E557, E593, E657

Spheres—3174, 3331, 3592, 3722, 3752, 3767, 3770, 3785, 3804, 3808, 3812, 3819, 3853, 3862, 3895, 3936, 3938, 3941, 3952, 3953, 3961, 3989, 4040, 4109, 4256, 4316, E188, E229, E264, E309, E337, E382, E530, E543, E583, E605, E633, E646, E900, E918

Tetrahedra, tetrahedra and associated spheres—3187, 3188, 3222, 3232, 3242, 3378, 3429, 3482, 3491, 3505, 3512, 3572, 3598, 3611, 3618, 3624, 3630, 3633, 3644, 3659, 3671, 3697, 3699, 3704, 3752, 3755, 3757, 3785, 3786, 3811, 3821, 3841, 3842, 3843, 3852, 3860, *3879*, 3888, 3934, 3946, 3947, 3959, 3972, 3978, 3981, 3987, 3988, 3998, 4001, 4004, 4005, 4011, 4013, 4015, 4018, 4024, 4039, 4044, 4049, 4069, 4075, 4090, 4106, 4116, 4127, 4131, 4149, 4150, 4153, 4160, 4175, 4178, 4187, 4192, 4197, 4201, 4204, 4206, 4208, 4213, 4218, 4219, 4224, 4228, 4233, 4234, 4251, 4265, 4266, 4271, 4279, 4297, 4354, *4366*, 4376, 4403, 4407, 4417, E86, E90, E141, E174, E245, E247, E318, E394, E437, E473, E477, E483, E493, E503, E507, E523, E540, E563, E573, E577, E587, E593, E607, E617, E623, E637, E667, E677, E715, E738, E763, E778, E795, E805, E830, E855, E882, E885, E894, E914, E928, E940

Volumes—2852, 3548, E29, E677, E757, E758

Miscellaneous (polyhedra; etc.)—*4176*, E923, *E945;* 2895, 2960, 3223, 3537, *3565*, 3569, 3575, 3595, E383, E407, E533, E889, E895

Plane analytic geometry

Assorted coordinate systems—3809, 3857, 3866, 4225, 4260, E241, *E285*, E699

Barycentric coordinates—3269, *3296*, 3335, 3392, 4414, E434, E495

Circles—2802, 3055, 3070, 3239, 3323, 3477, 3521, 3866, 3925, 3979, 4313, E62, E104, E226, E233, E762, *E832*, E939

Conics—2767, 2795, *2866*, 2873, 2898, *2892*, 2971, 3027, 3052, 3062, 3083, 3129, 3165, 3171, 3199, 3239, 3362, 3490, 3532, 3535, 3662, 3676, *3743*, *3805*, 3807, 3845, 3897, 3898, 3912, *3923*, *4019*, 4029, *4085*, 4225, 4274, 4393, E156, E191, E212, E403, E556, E645, E691, *E753*, E769

Envelopes—2891, 2981, 3288, 3293, 3310, 3361, 3559, 3669, 3813, E16, E569

Loci (conical)—2684, 2694, 2702, 2712, 2756, 2901, 2902, 2976, 2979, 2934, 3001, 3101, 3131, 3437, 3496, 3526, 3527, 4232, E98, E348

Loci (non-conical)—2865, 2887, 3042, 3160, 3195, 3292, 3483, 3680, 3694, 3744, 3745, 3760, 3774, 3962, 4147, 4200, 4361, E41, E303, E392, E471, E936

Polar coordinates—2919, 2923, 3037, *3179*, 3190, 3315, 3339, 3493, 3585, 3774, 3962, E41, E85, E311, E569, E653, E784, E835

Triangles—2834, 3254, 3315, 3387, 3470, 3485, 3576, 3637, 3712, 3818, 3839, 3925, *3967*, 3975, 4081, 4166, E131, E423, E831, E933

Trilinear coordinates—2684, 2715, 2834, 2902, 3024, 3132, 3268, *3440*, 3458,

3795, 3797, 3850, 3896, *3990*, 4059, 4154, 4170, 4193, 4200, 4398, E350

Miscellaneous (areas; cubics; intersections; points and lines; poles and polars; etc.)—3225, E65, E669; 2929, 3195, E904; 3533, 4273; 3338, 3823, *E774;* 4167, E666; 2839, 3291, 3359, 4343, *E419*, E649, E866

Solid analytic geometry

Assorted coordinate systems—3253, 4018, 4223, 4224, 4298, E443, E549, E800

Barycentric coordinates—3491, 3723, 3888, 3974, 3993, 4080, 4192, 4243, E370, E647, E885

Conicoids—2673, 2918, 3422, 3869, 4094, E175, E804, E843

Loci—2893, 2967, *3040*, 3041, 3224, 3311, 3407, *3586*, 3646, 3670, E74, E114, E200, E219, E260, E277, E314, E513, E868

Spheres—2897, 3174, 3544, 3941, 3953, 4015, 4064, E40, E553, E902

Tetrahedra—*3296*, 3560, 3639, 3644, 3689, 3759, 3840, 3849, 3986, 4015, 4248, E247, E252, E517

Miscellaneous (computations; surfaces; volumes; etc.)—3351, *E276;* 2992, 3773, E298; 3010, E145; 3370, 3619, 3627, 3992, *E481*, E820

Higher plane curves

Cardioid—2861, 3102, 3197, 3519, 3585, 3589, 3744, 3745, E569

Cubics—2824, 2928, 2978, 3444, *E413*, E501

Cycloids, epicycloids, hypocycloids—2691, 2863, 3221, 3235, 3533, 4115, 4181, E202, E600, E653, E731

Evolutes and involutes—2868, 3098, 3662, E110, E635, E936

Miscellaneous (caustic curves; cissoid; limaçon; quartics; spirals; strophoid; etc.)—2966, 3642; 3105, 3197; 2865, E471; 2977, 3160, 3884, E465; 3339, E41, E639, E784; 3483, 4026, E339, E479; 2917, 2923, 3100, 3110, 3302, 3566, 3813, 3962, *3999*, 4099, E243, E303, E392, *E753*, E934

Synthetic treatment of conics and conicoids—2904, 3215, 3230, 3237, 3247, 3288, 3356, 3384, 3415, 3496, 3822, 3861, 3869, 3898, *4019*, *4085*, 4112, E384, E447, E739, E797

Euclidean constructions

Approximate constructions—2932, 3291, E6, E475, E697

Approximate trisection of an angle—2972, 3114, 3490, 3522, 3525, 3563, 4134, E485

Circles—2734, 2753, 2768, 3262, 3298, 3453, E293, E296, E342, E457, E840

Compasses alone—3000, 3327, 3706, E100, E567

Conics—2789, 3617, 3650, 3728, 3732

Impossible constructions—2706, 2963, 3059, 3324, 3686, E574

Polygons—2745, 3128, 3134, 3192, 3219, E480, E529, E627, E713

Quadrilaterals—2667, 2682, 3069, 3319, 3368, *3417*, 3443, 3489, 3878, E8, E197, E279, E381, E707

Space constructions—2895, 2962, *2994*, 3768, 3798, E266, E421, E637, E693, E709, E749, E778, E865

Triangles—2687, 2750, 2796, 2820, 2903, 2910, 2920, 2962, 3029, 3119, 3178, 3240, 3248, 3320, 3324, 3336, 3357, 3369, 3374, 3405, 3410, 3441, 3520,

3558, 3668, 3681, 4026, 4076, E47, E108, E125, E134, E161, E170, E194, E239, E254, E257, E263, E329, E363, E441, E899, E933

Miscellaneous (straightedge alone; etc.)—*3089*, *3137*, E539, *E793;* 2885, 3170, 3381, 3904, E133, E196, E199, E203, E354, E396, E405, E698

Dissections

Planar—2799, *2933*, 3028, 3048, 3142, 3244, E80, E199, *E400*, E401, *E468*, E476, E541, E721, *E780*, E922

Spatial—*E4*, E5, *E554*, E724, E841, *E860*, *E888*

Geometry of complex numbers—2742, 2785, 2793, 2863, 3213, *3367*, 3444, 3524, *3547*, 3560, 3631, *3658*, 3712, *3718*, 3789, 3831, 3911, 3928, 3929, 3968, 3969, 3970, 3971, 3984, 3991, 4007, 4025, 4034, 4043, 4074, 4076, 4115, 4165, 4181, 4194, 4195, 4214, *4229*, *4239*, *4258*, 4301, 4338, 4364, 4370, E68, E698

Geometry of n dimensions—3656, *3683*, 3715, 3729, 3742, 3821, 3830, 3840, 3843, 3846, 3847, *3863*, 3888, 3908, *3909*, 3963, 3968, 3978, 4049, 4079, 4116, 4126, *4222*, E40, E233, E252, *E740*, E772, E773, E820

Synthetic projective geometry

Conics—2756, *2892*, 3003, 3138, 3156, 3247, *3565*, 3650, 3724, 3807, *4019*, 4162, 4193, 4334, E659, E739

Projective generalization—2921, 2988, 3270, 3660, E501, E571

Triangles—3202, 3205, 3602, *3957*, 3976, 4102, *E481*, E795

Miscellaneous (constructions; envelopes; orthogonal projection; space problems; etc.)—2885, 3119, 3240, 3617; 3035, 3039, 3156, 3473; 3474, *3565;* 3993, 4061, 4141, E453; 2977, 3041, 3436, 3698, 3878, 4087, E289, E661, E675

Analytic projective geometry

Miscellaneous (conics; envelopes; stereographic projection; etc.)—2924, 4063, *E481;* 3030, 3039, 3268, 3473; 2897, 3722; 3736, 4079, E675

Plane differential geometry

Curves—2691, 2855, 2950, 2990, 3078, 3265, 3372, 3451, 4280, E600, *E610*

Envelopes—2733, 2763, 2819, 2827, 2861, 2890, 3415, 3416, 4245

Tangents, normals, curvature; involutes—2824, 2957, 2966, 3110, 3200, 3217, 3469, *3999*, 4036, 4292; 3098, 3551, E810

Differential geometry of space

Space curves—3019, 3238, 3398, 3447, 3562, 3613, 3777, 3906, 4105, 4128, 4238, E551

Surfaces—2781, 2948, 2955, 2968, 3120, 3317, 3340, 3597, 3992, 4028, 4033, E836

Topology—3017, 3432, *3829*, *4361*, *E555*, *E711*, *E788*, E923

Unsolved geometry problems—2721, 2762, 2816, 2875, 2888, 2946, 2964, 3004, 3012, 3033, 3060, 3077, 3087, 3112, 3147, 3158*, 3163*, 3180*, 3186*, 3243, 3279, 3355, 3380, 3382, 3393, 3394, 3456, 3466, 3509, 3511, 3540, 3554, 3568, 3570, 3574, 3578, 3600, 3604, 3610, 3623, 3648, 3749, 3782, 3787, 3806, 3858, 3859, 3887, 3905, 3966, 4032, 4035, 4077, 4096, 4113, 4129, 4145, 4241, 4306, 4359, 4368, 4369, E570, E585

ANALYSIS

Trigonometry

Analytical trigonometry—2772, 2828, *3011*, 3044, *4152*, E26, E301, E620

Areas—3002, E60, E70, E183, *E681*, E847, E892

Equations—2912, 2973, 3241, *4136*, *4400*, E149, E157, E209

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Identities—2697, 2995, 2997, 3181, 3245, 3563, E48, E72, E82, E167, E201, E256, E286, E292, E302, E863, E871

Polygons—2780, 2853, 2925, 2931, 2932, 2965, 3014, 3401, E673

Space problems—3307, 3503, 3504, 3546, E29, E748

Spherical trigonometry—2850, 2897, 3022, 3130, 3136, E741

Triangles—2728, 2747, 2761, 2896, 3054, 3470, 3576, 3602, 3733, *3957*, *4185*, E15, E44, E121, E177, E181, E194, E239, E301, E302, E305, *E312*, E315, E439, E688, E701, E913

Miscellaneous (computation; inequalities; physics; etc.)—2696, E87, E103; *3740*, 4082, E723; 2716, 2876; 2989, E120

Differential calculus

Curvature, curvilinear motion, etc.—2855, 2862, 2872, *3999*, E45, E675, E691, E904

Curves—3061, 3168, 3265, *E817*, E838

Differentials and differentiation—2674, 2941, 3161, 3487, 3599, 3714, 4246, 4397, E53, E789, *E846*

Envelopes—2819, 2890, 3415, 3416, 3608, 3669, 3694, 3750, 4154, 4182, E84

Maxima and minima—2677, 2689, 2690, 2748, 2769, 2818, *2871*, *2936*, 2954, 3067, 3102, 3134, 3284, 3318, 3567, *3713*, 3913, *4336*, E3, E45, E81, E92, E138, E304, E325, *E436*, E489, E511, E558, E574, *E687*, E731, E781, E814, E858, E862, E944

Maxima and minima (more than one variable)—2680, 2731, 2742, 2773, 2843, 2860, *2980*, 3301, 3365, 3537, 3754, *4053*, E18, E96, E287, E729

Tangents and normals—2824, 3100, 3519, 3617, 4151, E45, E76

Miscellaneous (identities; rates; etc.)—3388, 4156; *2936*, 3493, E169, E316; 2719, 3168

Integral calculus

Arc lengths—2868, 3008, 3079, 3088, 3105, *4172*, E630, E665, E679, E826

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Integration—3084, 3590, 3943, 4357, E32, E864

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Surface areas—2771, 2813, 2949, 3149, 3151, 3412, 3413, E138, E559

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Miscellaneous (average values; centroids; elliptic integrals; fluid force, etc.)—3501, 3502, 4262, E270; 2683, 3155, *4336*, E875; 3557, *3914;* 3123, E807; *3438*, 3464, 4285, E88, *E275*

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Euler's constant—*4045*, *4046*, *4286*, *E819*, E938

Infinite sequences—*3177*, 3478, *4330*, *4332*, 4333, *4341*, *4358*, *4363*, 4405, E474, E498, *E598*, E835, E874

Limits—2709, 2794, *2845*, 2857, 2986, 3013, *3053*, 3313, 3334, 3434, 3478, 3516, 3571, *3612*, 3675, 3676, 3693, 3977, 4151, *4186*, *4198*, *4255*, *4290*, 4324, *4372*, 4374, 4378, E216, E236, E241, E244, E450, E454, E474, E504, E528, *E598*, E694, E790, *E853*, E867, E874, E910, E924

Miscellaneous (continuity; differentiation; Euler's theorem; integral inequalities; multiple integration; etc.)—*4216*, E829; 3108, 3159, E770, E935; 2812, 3601; 2726, 2999, *3104*, *4264; 3408*, 3460; *3043*, 3557, 4000, *4371*

Infinite series

Convergence and divergence—*2752*, 2831, 3260, *3448*, 3794, *4142*, *4250*, *4278*, 4374, *4391*, 4409, *4415*, E615, *E737*, E809, *E824*, E867

Expansion of functions—2737, *3045*, *3778*, *4067*, *4402*, *4418*, E376

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Summation—2675, 2701, 2777, 2907, 2913, *2991*, 3025, 3047, 3068, 3071, *3051*, 3337, *3343*, *3348*, 3571, 3579, 3647, *3737*, *3801*, *3996*, *4045*, *4046*, *4138*, 4199, 4242, 4253, *4293*, *4305*, *4318*, *4321*, *4346*, *4353*, 4357, *4382*, 4384, E109, E618, E664, E704, E810, E837, E844, *E854*, E881, E893, E908

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First order—2950, 3193, 3272, 3275, 3282, 3442, 3486, 3500, 3531, *3778*, 3924, *4315*

Geometry—3078, 3098, 3200, 3217, 3286, 3654, 4006, 4037, 4041, E586, E934

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Partial differential equations—3333, 3588, 3601, *3826*, 3995, 4158, E645

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* Numbers with an asterisk are *corrected* numbers. To rectify two errors in the original numbering in the MONTHLY, all numbers of proposed problems from 3144 [1925, 433] to 3180 [1926, 159] have been increased by four, and all numbers of proposed problems from 3173 [1926, 283] to 3193 [1926, 338] have been increased by twelve. The problems proposed in the Aug.-Sept. 1926 issue begin with 3206 as they should.

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